

RICCI FLOW UNSTABLE CELL CENTERED AT AN EINSTEIN METRIC ON THE TWISTOR SPACE OF POSITIVE QUATERNION KÄHLER MANIFOLDS OF DIMENSION ≥ 8

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ABSTRACT. We construct a 2-parameter family \mathcal{F}^Z of Riemannian metrics on the twistor space \mathcal{Z} of a positive quaternion Kähler manifold M satisfying the following properties : (1) the family \mathcal{F}^Z contains an Einstein metric g^Z and its scalings, (2) the family \mathcal{F}^Z is closed under the operation of making the convex sums, (3) the Ricci map $g \mapsto \text{Ric}(g)$ defines a dynamical system on the family \mathcal{F}^Z , (4) the Ricci flow starting at any metric in the family \mathcal{F}^Z stays in \mathcal{F}^Z and is an ancient solution having the Einstein metric g^Z as its asymptotic soliton.

This means that the family \mathcal{F}^Z is a 2-dimensional “unstable cell” w.r.to the Ricci flow which is “centered” at the Einstein metric g^Z . We apply the estimates for the covariant derivative of the curvature tensor under the Ricci flow to this “unstable cell” and settle the LeBrun-Salamon conjecture : any irreducible positive quaternion Kähler manifold is isometric to one of the Wolf spaces.

§1. Introduction.

In this paper, we construct a new Einstein metric together with a two parameter family of metrics containing it on the twistor space of a positive quaternion Kähler manifold and describe the behavior of this family under the Ricci flow.

Because the background of this attempt is rather heavy, we start with the explanation what “Ricci flow unstable cell” means. In his beautiful paper [P], G. Perelman introduced his \mathcal{W} -functional

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV$$

where M is an n -dimensional smooth (closed) manifold, g_{ij} is a Riemannian metric on M , f is a smooth function on M , dV is the Riemannian volume form of the metric g_{ij} and τ is a positive scaling parameter. The remarkable property of the \mathcal{W} -functional proved in [P] is the following. Consider the system

$$\begin{cases} \partial_t g_{ij}(t, x) = -2R_{ij}(g(t, x)) \\ \partial_t u(t, x) = -\Delta_x u(t, x) + Ru(t, x) & (u := (4\pi\tau)^{-\frac{n}{2}} e^{-f}) \\ \partial_t \tau(t) = -1 \end{cases}$$

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of evolution equations on M , where Δ is the Laplacian w.r.to the metric $g_{ij}(t, x)$ acting on functions and R is the scalar curvature of the metric $g_{ij}(t, x)$. Note that the second equation is the conjugate heat equation and therefore $\int_M u dV$ is independent of t (usually normalized to be 1). The short time existence of this system of evolution equations is a consequence of the short time existence of the Ricci flow equation (see [P, §1]). The \mathcal{W} -functional is monotone non-decreasing along its solution. In fact we have the monotonicity formula

$$\partial_t \mathcal{W} = 2 \int_M \tau |R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}|^2 u dV .$$

The origin of this monotonicity formula is the following. If we apply the family of t -dependent diffeomorphisms $\{\phi_t\}$ obtained by diagonally integrating the t -dependent vector field $-\nabla f$ (∇ being the Levi-Civita connection w.r.to the metric $g_{ij}(t, x)$) to the above system of equations, we get the system

$$\begin{cases} \partial_t g_{ij}(t, x) = -2(R_{ij}(t, x) + \nabla_i \nabla_j f(t, x)) \\ \partial_t f(t, x) = -\Delta_x f(t, x) - R(t, x) + \frac{n}{2\tau(t)} \\ \partial_t \tau(t) = -1 \end{cases}$$

which is precisely the L^2 -gradient flow of the \mathcal{W} -functional under the constraint that $(4\pi\tau)^{-\frac{n}{2}} e^{-f} dV$ is a fixed measure dm on M independent of t . In this situation, the \mathcal{W} -functional should be interpreted as

$$\mathcal{W}^m(g_{ij}, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n] dm .$$

In particular the symmetry under which the functional \mathcal{W}^m is invariant is the subgroup of $\text{Diff}_0(M)$ consisting of dm -preserving diffeomorphisms. The action of other diffeomorphisms is introduced as follows. For $\phi \in \text{Diff}_0(M)$ we define f^ϕ by setting $dm = (4\pi\tau)^{-\frac{n}{2}} e^{f^\phi} dV_{\phi^*g}$. If ϕ preserves dm then f^ϕ is just $\phi^* f$ and the converse is also true. The action of $\text{Diff}(M)_0$ is thus defined on the configuration space $\{(g_{ij}, f, \tau)\}$ on which the functional \mathcal{W}^m is defined.

Perelman's \mathcal{W} -functional is a “coupling” of the **logarithmic Sobolev functional**¹ and the **Hilbert-Einstein functional**². Suppose that there exists a critical point of the \mathcal{W}^m -functional, which corresponds to a Ricci soliton

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} = 0 .$$

This is interpreted, at time $t = -1$ ($\tau = 1$), as the initial condition for the Ricci flow equation (the solution satisfies the above equation of the Ricci soliton, which evolves under a 1-parameter group

¹ The logarithmic Sobolev inequality on the n -dimensional Euclidean space \mathbb{R}^n is the following. Let $f = f(x)$ satisfies the constraint $\int_{\mathbb{R}^n} (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_{\text{euc}} = 1$. Then we have

$$\int_{\mathbb{R}^n} [\tau |\nabla f|^2 + f - n] (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_{\text{euc}} \geq 0$$

where the equality holds iff $f(x) = \frac{|x|^2}{4\tau}$.

² The Hilbert-Einstein functional is $\int_M R dV_g$ for a closed Riemannian manifold (M, g) and the critical points are Einstein metrics.

of diffeomorphisms of M generated by the “gradient” vector field ∇f). Perelman [P] showed that the Ricci soliton is characterized by the equality case of the logarithmic Sobolev inequality in the following way. Let $g_{ij}(-1)$ satisfy the above equation at time $t = -1$ and $g_{ij}(t)$ the corresponding solution of the Ricci flow, i.e., the Ricci soliton with initial metric $g_{ij}(-1)$. Then the logarithmic Sobolev inequality on $(M, g_{ij}(t))$ introduced in [P] is

$$\begin{aligned} \mathcal{W}(g_{ij}(t), \tilde{f}, -t) &\geq \mathcal{W}(g_{ij}(t), f(t), -t) = \inf_{\tilde{f}: \int_M (4\pi(-t))^{-\frac{n}{2}} e^{-\tilde{f}} dV_{g(t)} = 1} \mathcal{W}(g_{ij}(t), \tilde{f}, -t) \\ &=: \mu(g_{ij}(t), -t) = \mu(g_{ij}(-1), 1) \end{aligned}$$

where \tilde{f} is any smooth function on M satisfying $\int_M (4\pi(-t))^{-\frac{n}{2}} e^{-\tilde{f}} dV_{g(t)} = 1$. This observation gives us an important information on the behavior of the \mathcal{W} -functional at a critical point (i.e., the Ricci soliton). We look at the Hessian of the \mathcal{W}^m -functional at the critical point. The \mathcal{W}^m -functional is invariant under the group of all dm -preserving diffeomorphisms and therefore this action corresponds to the zeros of the Hessian. On the other hand, the action of the diffeomorphisms which do not preserve dm may be given by the following way. Let ϕ be such a diffeomorphism. Introduce f^ϕ by setting $dm = (4\pi\tau)^{-\frac{n}{2}} e^{-f^\phi} dV_{\phi^*g}$ and define $\phi^*(g, f, \tau) = (\phi^*g, f^\phi, \tau)$. Then we have

$$\mathcal{W}^m(\phi^*(g, f, \tau)) = \int_M [\tau(R_{\phi^*g} + |\nabla f^\phi|_{\phi^*g}^2) + f^\phi - n] \underbrace{(4\pi\tau)^{-\frac{n}{2}} e^{-f^\phi} dV_{\phi^*g}}_{dm}$$

and therefore the \mathcal{W}^m -functional increases in the direction of the action of the diffeomorphisms which do not preserve dm , which follows from the logarithmic Sobolev characterization of the Ricci soliton. This implies that the tangent space of the configuration space $\{(g, f, \tau)\}$ of the functional \mathcal{W}^m decomposes into three subspaces V_0 , V_+ and V_- . Here, V_0 corresponds to the action of the dm -preserving diffeomorphisms (Hess = 0), V_+ corresponds to the action of the diffeomorphisms which do not preserve dm (Hess ≥ 0) and finally V_- (Hess ≤ 0) corresponds to the rest³.

The above discussion means that we can speak of the stable/unstable cell w.r.to the Ricci flow in the space of Riemannian metrics on a given manifold (with respect to the L^2 -gradient flow of the functional \mathcal{W}^m).

Perelman announced the following convergence theorem of the Kähler Ricci flow $\partial_t g_{i\bar{j}}(t, x) = -R_{i\bar{j}}(g(t, x)) + g_{i\bar{j}}(t, x)$ with the initial Kähler metric $g_{i\bar{j}}(0, x)$ representing $c_1(M)$ where M is a Fano manifold M (see Tian-Zhu [T-Z]).

Theorem 1.1 (Perelman, Tian-Zhu). *If M admits a Kähler-Einstein metric, then, for any initial Kähler metric $g_{i\bar{j}}(0, x)$ representing $c_1(M)$, the solution to the Kähler-Ricci flow converges to a Kähler-Einstein metric in the sense of Cheeger-Gromov.*

This implies that the normalized Kähler-Ricci flow starting at a Kähler metric in $c_1(M)$ on a Fano manifold admitting a Kähler-Einstein metric forms a stable cell w.r.to the dynamical system defined by the Ricci flow solution (in short, we call this a *Ricci flow stable cell*).

It is then interesting to construct an example of a *Ricci flow unstable cell* centered at an Einstein metric on an Einstein manifold. Theorem 1.1 implies that, on a positive Kähler-Einstein manifold with $b_2 = 1$, one can find a Ricci flow unstable cell centered at a Kähler-Einstein metric (if any) only from a solutions of the non Kähler Ricci flow, or one should find another non-Kähler Einstein metric together with a Ricci flow unstable cell.

³ This is very similar to the behavior of the Hilbert-Einstein functional under the Yamabe problem.

This paper is an attempt toward this direction. A Ricci flow unstable cell centered at an Einstein metric (if any) necessarily consists of *ancient solutions* (a Ricci flow solution is said to be an ancient solution [H] if it is defined on a time interval $(-\infty, T]$ where $T \in \mathbb{R}$) whose asymptotic soliton (in the sense of [P, §11]) is the Einstein metric. Because there is no guarantee for evolution equations (such as the Ricci flow equation) to be solved in the past direction even locally, the existence of the ancient solution must be a miracle which comes from a very special geometric situation. It is best explained by Perelman's result : ancient solutions appear from the rescaling procedure of the singularities developed in the Ricci flow in finite time (this is a consequence of Perelman's Local Non-Collapsing Theorem [P, §4 and §7]). In this paper we pick up the *non holomorphic twistor fibration of the twistor space \mathcal{Z} of positive quaternion Kähler manifolds M* as a candidate of a special geometric situation admitting a Ricci flow unstable cell centered at an Einstein metric. The strategy is to think of the non holomorphic \mathbb{P}^1 -fibration $\pi : \mathcal{Z} \rightarrow M$, imagine the collapse of \mathcal{Z} to M (i.e., the \mathbb{P}^1 -fiber collapses) or to \mathbb{P}^1 (i.e., the base M collapses) as a “singularity” developed in the Ricci flow on \mathcal{Z} in finite time and then try to construct the corresponding “ancient solutions” by suitable rescaling procedure. However, this attempt do not seem to work well if one stick to the collapse along the twistor fibration. In §2, we study the family \mathcal{F}^{can} of the canonical deformation metrics and show that \mathcal{F}^{can} is “stable” under the Ricci flow. It turns out that the Ricci flow solution converges to the Kähler-Einstein metric if we start the Ricci flow at any canonical deformation metric sufficiently close to the Kähler-Einstein metric. The purpose of §2 is to explain why the canonical deformation metrics do not form the Ricci flow unstable cell centered at the Kähler-Einstein metric. In §3, we introduce an idea to “kill” the basic reason why the family of the canonical deformation metric is not a Ricci flow unstable cell and modify the construction of the canonical deformation metric. As a result, we construct a new Einstein metric $g^{\mathcal{Z}}_{\sqrt{\frac{1}{n+2}}}$ and a new family of $\mathcal{F}^{\mathcal{Z}}$ metrics (containing $g^{\mathcal{Z}}_{\sqrt{\frac{1}{n+2}}}$) on the twistor space and show that the family $\mathcal{F}^{\mathcal{Z}}$ constitutes a Ricci flow unstable cell centered at the Einstein metric $g^{\mathcal{Z}}_{\sqrt{\frac{1}{n+2}}}$.

An oriented $(4n)$ -dimensional Riemannian manifold (M^{4n}, g) is said to be a *quaternion Kähler manifold*, if its holonomy group is contained in $\text{Sp}(n)\text{Sp}(1)$ ($\subset \text{SO}(4n)$) (for a precise definition, see §2). A locally irreducible (in the sense of the local de Rham decomposition defined from the irreducible decomposition of the tangent space under the action of the local holonomy group) quaternion Kähler manifold is necessarily Einstein. Therefore we can classify locally irreducible quaternion Kähler manifolds by the sign of the scalar curvature into three classes. A complete locally irreducible quaternion Kähler manifold is called positive (resp. locally hyper-Kähler, negative), if its scalar curvature is positive (resp. zero, negative). The locally hyper-Kähler property is equivalent to the absence of the $\text{Sp}(1)$ -component in the $\text{Sp}(n)\text{Sp}(1)$ -holonomy. An irreducible positive quaternion Kähler manifold turns out to be a simply connected positive Einstein manifold. From here on, we restrict our attention to locally irreducible quaternion Kähler manifolds (and therefore we just say “quaternion Kähler” for “locally irreducible quaternion Kähler”).

The principal bundle of oriented orthonormal frames of a quaternion Kähler manifold is reduced to a principal $\text{Sp}(n)\text{Sp}(1)$ -bundle \mathcal{P} (the holonomy reduction). Associated to a quaternion Kähler manifold is the twistor space \mathcal{Z} , which is constructed from \mathcal{P} and the $\text{Sp}(1)$ -part of the holonomy group $\text{Sp}(n)\text{Sp}(1)$ by putting

$$\mathcal{Z} := \mathcal{P} \times_{\text{Sp}(n)\text{Sp}(1)} \mathbb{P}^1 = \mathcal{P}/\text{Sp}(n)\text{Sp}(1) \cap U(2n)$$

(this is a \mathbb{P}^1 -bundle over M). The horizontal distribution w.r.to the Levi-Civita connection on the twistor fibration $\pi : \mathcal{Z} \rightarrow M$ is canonically defined and thus the twistor space \mathcal{Z} is equipped with

a natural almost complex structure (which turns out to be integrable) and two kinds of family of metrics which is “compatible” with the twistor space construction. The first one is the family of the *canonical deformation metrics* defined by adding scaled standard metrics of the \mathbb{P}^1 -fiber and the fixed base metric on M by using the horizontal distribution (we will use the description by Chow and Yang in [C-Y]). The second one (called the family of *Z-metrics*) will be introduced in this paper (see §3), whose construction is based on the canonically defined a horizontal $(4n - 2)$ -dimensional distribution \mathcal{D}' and the “associated” non-horizontal $(4n)$ -dimensional distribution \mathcal{D} on \mathcal{Z} . The goal of this paper is to describe the behavior under the Ricci flow of the family of Z-metrics (and comparison with that of the family of canonical deformation metrics).

Typical examples of positive quaternion Kähler manifolds are the “Wolf spaces” [W], i.e., $(4n)$ -dimensional positive quaternion Kähler symmetric spaces. Wolf [W] proved that for each compact simple Lie group G , there is a Wolf space G/H . The classical ones are

$$\begin{aligned} \mathbb{HP}^n &= \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n) \times \mathrm{Sp}(1)}, \quad \mathrm{Gr}_2(\mathbb{C}^n) = \frac{\mathrm{SU}(n)}{\mathrm{S}(\mathrm{U}(n-2) \times \mathrm{U}(2))}, \\ \widetilde{\mathrm{Gr}}_4(\mathbb{R}^n) &= \frac{\mathrm{SO}(n)}{\mathrm{SO}(n-4) \times \mathrm{SO}(4)} \end{aligned}$$

and there are exceptional cases. The noncompact dual of the Wolf spaces are examples of negative quaternion Kähler manifolds. There exist many other examples of noncompact negative quaternion Kähler manifolds which are not symmetric (e.g., Alexeevskii [A], Galicki [G]). Moreover, we remark that Galicki-Lawson’s quaternion Kähler reduction method ([G-L]) produces many examples of positive quaternion Kähler orbifolds which are not symmetric⁴.

Conjecture 1.2 ([L-S]). *Any irreducible positive quaternion Kähler manifold is isometric to one of the Wolf spaces.*

The twistor space of a Wolf space is a generalized flag manifold (rational homogeneous manifold) of the form G/P , which is a Fano manifold. More precisely, the twistor space of a positive quaternion Kähler manifold is a “contact Fano manifold” (see (2-13)). LeBrun [LeB] proved that a contact Fano manifold \mathcal{Z} is realized as the twistor space of some positive quaternion Kähler manifold M if and only if \mathcal{Z} admits a Kähler-Einstein metric.

Conjecture 1.3 ([L-S]). *Any contact Fano manifold admits a Kähler-Einstein metric.*

The affirmative answers to Conjectures 1.2 and 1.3 combined with the result of [LeB] imply that a compact complex manifold \mathcal{Z} is the twistor space of a Wolf space if and only if \mathcal{Z} is a contact Fano manifold. On the other hand, using complex algebraic geometry of contact Fano manifold, LeBrun and Salamon [L-S] proved that there exist only finitely many positive quaternion Kähler manifolds with fixed dimension.

We give a brief description on the basic properties of the family of Z-metrics and their construction.

The family $\mathcal{F}^{\mathcal{Z}}$ of Z-metrics satisfies the following properties : (1) the family $\mathcal{F}^{\mathcal{Z}}$ contains an Einstein metric $g^{\mathcal{Z}}$ which is different from the well-known two Einstein metrics in the family of canonical deformation metrics, (2) the family $\mathcal{F}^{\mathcal{Z}}$ is closed under positive scalar multiples and summation, (3) the family $\mathcal{F}^{\mathcal{Z}}$ is stable under the Ricci map $g \mapsto \mathrm{Ric}(g)$ (i.e., the Ricci map sends

⁴ the arguments in §2 for the twistor space of positive quaternion Kähler manifolds cannot be generalized to the orbifold case (see Remark 3.5 in §3).

\mathcal{F}^Z to itself), (4) any Ricci flow solution starting with initial metric in the family \mathcal{F}^Z stays in \mathcal{F}^Z and is an ancient solution whose asymptotic soliton is the Einstein metric g^Z .

The construction of the family \mathcal{F}^Z is based on the canonically defined real $(4n)$ -dimensional non-horizontal distribution $\mathcal{D} = \{\mathcal{D}_z\}_{z \in \mathcal{Z}}$. The distribution \mathcal{D}' is constructed as follows (see §3). There canonically exists a horizontal $(4n - 2)$ -dimensional distribution $\mathcal{D}' = \{\mathcal{D}'_z\}_{z \in \mathcal{Z}}$. Here, \mathcal{D}'_z is $J(z)$ -invariant where $J(z)$ is the orthogonal complex structure corresponding to z . Let $L_{J(z)} = (\mathcal{D}'_z)^\perp_{\mathcal{H}_z}$ where $\mathcal{H} = \{\mathcal{H}_z\}_{z \in \mathcal{Z}}$ is the horizontal distribution. Let $L_{J(z)}$ be the “diagonal” in the complex 2-dimensional distribution $\{\mathcal{V}_z \oplus L'_{J(z)}\}_{z \in \mathcal{Z}}$ where \mathcal{V}_z is the vertical distribution. The $(4n)$ -dimensional non-horizontal distribution \mathcal{D} on the twistor space \mathcal{Z} is defined by $\mathcal{D}_z = \mathcal{D}'_z \oplus L'_{J(z)}$, which is still transversal to the twistor fibration $\mathcal{Z} \rightarrow M$. Set $\mathcal{D}^\perp = \{\mathcal{D}_z^\perp\}_{z \in \mathcal{Z}}$. This is a 2-dimensional distribution which is invariant w.r.to the canonical complex structure of the twistor space \mathcal{Z} . Here, the orthogonal complement of \mathcal{D}_z is taken w.r.to the basic canonical deformation metric g_1^{can} (see §2). The family of Z-metrics is defined in the following way. Let $\{\xi_i\}_{i=0}^3$ be a local frame of \mathcal{D} (here we are using the column n -vector notation identifying \mathcal{D}_z with $T_{\pi(z)}M = \mathbb{H}^n$, as in [CY]) satisfying a condition that $\{d\pi(\xi_i)\}_{i=0}^3$ gives an oriented orthonormal basis of $(T_{\pi(z)}M, g_{\pi(z)})$. Let $\{X^i\}_{i=0}^3$ denote the dual coframe which we extend to be zero on \mathcal{D}^\perp . Let $\{\alpha_1\}_{i=1}^3$ be the $\text{sp}(1)$ -part of the Levi-Civita connection form of the original quaternion Kähler metric and $\{\alpha_1, \alpha_3\}$ the orthonormal coframe dual to the orthonormal frame along the twistor line. $\cong \mathbb{P}^1 = \text{Sp}(1)/\text{U}(1)$. Set $\hat{\alpha}_j = \alpha_j - \sum_{i=0}^3 \alpha_1(\xi_i) X^i$ ($j = 1, 3$). The Z-metric with parameter λ is defined as declaring that $\{\lambda \hat{\alpha}_1, \lambda \hat{\alpha}_3, X^0, X^1, X^2, X^3\}$ forms an oriented orthonormal coframe :

$$g_\lambda^Z = \lambda^2(\hat{\alpha}_1^2 + \hat{\alpha}_3^2) + {}^tX^0 \cdot X^0 + {}^tX^0 \cdot X^1 + {}^tX^0 \cdot X^1 + {}^tX^2 \cdot X^2 + {}^tX^3 \cdot X^3.$$

For curvature computation we must choose a good frame satisfying certain jet conditions at one point, which we briefly describe in the following. A point $z \in \mathcal{Z}$ uniquely defines an orthogonal complex structure J on $\mathcal{H}_z \cong (T_m M, g_m)$ and its real $(2n)$ -dimensional J -complex subspace spanned by ξ_0 and $J\xi_0$, where ξ_0 is determined uniquely modulo $\text{SO}(2)$ -rotation and should be understood as a column n vector (and therefore $\{\xi_0, I\xi_0, J\xi_0, K\xi_0\}_{\text{span}}$ is $(4n)$ -dimensional)⁵. We extend ξ_0 to a “unit length” germ at $m \in M$ so that $\nabla \xi_0$ satisfies certain condition. Moreover we extend J to a section germ of \mathcal{S} so that $\nabla J = 0$ at m , where \mathcal{S} is the 3-dimensional subbundle of $\text{End}^{\text{skew}}(TM)$ defining the quaternion Kähler structure of M . Next, we extend I and K to section germs of \mathcal{S} so that (i) $\{I, J, K\}$ constitutes the oriented orthonormal frame germ of \mathcal{S} (so $\{I, J, K\}$ satisfies the quaternion relations) and (ii) the oriented orthonormal frame germ σ of the twistor fibration $\mathcal{Z} \rightarrow M$ defined by $\{\xi_0, I\xi_0, J\xi_0, K\xi_0\}$ satisfies the condition

$$(d\sigma)_m(T_m M) = \mathcal{D}_z$$

(of course this is defined by a section germ of $\mathcal{P} \rightarrow M$ composed with the projection $\mathcal{P} \rightarrow \mathcal{Z}$). Note that such a pair $\{I, K\}$ is defined uniquely modulo $\text{SO}(2)$ -rotation. This is the “condition” which should be satisfied by I and K in the construction of our Z-metrics⁶. Let the triple $\{\alpha_i\}_{i=1}^3$

⁵ The expression ξ^i (resp. its dual X^i) for $i = 0, 1, 2, 3$ should be understood as row (resp. column) n vectors if they are used in the quaternion Kähler context related to the base manifold M (e.g. in such a notation $\sum_{i=0}^3 {}^tX^i \cdot X^i$ which represents the quaternion Kähler metric on M) or just a 1-form obtained by taking the sum in other occasion (e.g., in such a notation $\alpha_1 - \alpha_1(\xi_1)X^1 - \alpha_1(\xi_3)X^3$ which represents a certain modification of the $\text{sp}(1)$ -part of the connection form). For this notation is used in [C-Y]. See §2 of this paper.

⁶ Important comparison : The construction of Z-metrics is characterized by the condition $(d\sigma)_m(T_m M) = \mathcal{D}_z$. On the other hand the construction of the canonical deformation metrics is characterized by the condition $\nabla I = 0$ and $\nabla K = 0$ at m .

be the $\mathfrak{sp}(1)$ -part of the connection form defined on the holonomy reduction \mathcal{P} of the oriented orthonormal frames. Let the orthogonal complex structure J of $T_m M$ defined by an element of \mathcal{S}_m represented by $(0, 1, 0)$. The pair $\{\alpha_1, \alpha_3\}$ then represents the infinitesimal deformation of the orthogonal complex structures J at $z \in \mathcal{Z}$ and defines the induced horizontal subspace at z . Set $\xi_1 = I\xi_0$, $\xi_2 = J\xi_0$, $\xi_3 = K\xi_0$. Let $\{X^0, X^1, X^2, X^3\}$ be the coframe field dual to $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ (understood as canonical 1-forms defined on \mathcal{P}). The family \mathcal{F}^Z of metrics on \mathcal{Z} is defined by declaring that

$$\begin{aligned} \hat{\alpha}_1 &:= \underbrace{\alpha_1 - \alpha_1(\xi_0)X^0 - \alpha_1(\xi_1)X^1 - \alpha_1(\xi_2)X^2 - \alpha_1(\xi_3)X^3}_{\text{annihilates } \mathcal{D}_z}, \\ \hat{\alpha}_3 &:= \underbrace{\alpha_3 - \alpha_3(\xi_0)X^0 - \alpha_3(\xi_1)X^1 - \alpha_3(\xi_2)X^2 - \alpha_3(\xi_3)X^3}_{\text{annihilates } \mathcal{D}_z}, \\ &\quad \underbrace{X^i \ (i = 0, 1, 2, 3)}_{\text{dual to orthonormal basis of } \mathcal{D}_z} \end{aligned}$$

forms an oriented orthonormal frame and therefore the Z -metric g_λ^Z is defined as

$$\begin{aligned} \rho g_\lambda^Z &:= \rho \left[\lambda^2 \left\{ \left(\underbrace{\alpha_1 - \alpha_1(\xi_0)X^0 - \alpha_1(\xi_1)X^1 - \alpha_1(\xi_2)X^2 - \alpha_1(\xi_3)X^3}_{\text{invisible}} \right)^2 \right. \right. \\ &\quad \left. \left. + \left(\underbrace{\alpha_3 - \alpha_3(\xi_0)X^0 - \alpha_3(\xi_1)X^1 - \alpha_3(\xi_2)X^2 - \alpha_3(\xi_3)X^3}_{\text{invisible}} \right)^2 \right\} \right. \\ &\quad \left. + {}^t X^0 \cdot X^0 + {}^t X^1 \cdot X^1 + {}^t X^2 \cdot X^2 + {}^t X^3 \cdot X^3 \right] \end{aligned}$$

which is expressed in the orthonormal coframe germ at $z \in \mathcal{Z}$ (one point)⁷.

We show that the subspace \mathcal{F}^Z is foliated by the trajectories of the Ricci flow solution, i.e., the Ricci flow whose initial metric belongs to the subspace \mathcal{F}^Z stays in \mathcal{F}^Z as long as the solution exists and is an ancient solution in the sense of Hamilton [H]. The trajectory consisting of positive scalar multiples of the Einstein metric $g_{\sqrt{\frac{1}{n+2}}}^Z$ is the asymptotic soliton of all other Ricci flow trajectories in \mathcal{F}^Z . In particular, the Einstein metric $g_{\sqrt{\frac{1}{n+2}}}^Z$ is an “unstable fixed point” under the dynamical system defined by the trajectories of the Ricci flow solutions. The Ricci flow solution starting at a metric in \mathcal{F}^Z with $\lambda^2 > \frac{1}{n+2}$ Gromov-Hausdorff converges (after appropriate scalings) in finite time to the Carnot-Carathéodory metric defined on the $(4n)$ -dimensional distribution \mathcal{D} in \mathcal{Z} , which isometrically projects to the original quaternion Kähler metric on the base manifold M . We then apply Bando-Shi estimate ([B], [Sh1,2]) for ∇Rm under the Ricci flow to these ancient solutions to prove a limit formula which implies the LeBrun-Salamon conjecture that any irreducible positive quaternion Kähler manifold (M, g) is isometric to one of the Wolf spaces. The technical part of the proofs of all results in this paper is based on the moving frame computation on the twistor

⁷ Although two 1-forms $\alpha_1 - \alpha_1(\xi_0)X^0 - \alpha_1(\xi_1)X^1 - \alpha_1(\xi_2)X^2 - \alpha_1(\xi_3)X^3$ and $\alpha_3 - \alpha_3(\xi_0)X^0 - \alpha_3(\xi_1)X^1 - \alpha_3(\xi_2)X^2 - \alpha_3(\xi_3)X^3$ are non orthogonal w.r.to the canonical deformation metric, we declare that they are orthogonal w.r.to the Z -metrics at $z \in \mathcal{Z}$. Moreover, although the terms $\alpha_1(\xi_0)X^0$ and so on are “invisible” at $z \in \mathcal{Z}$ we cannot ignore them because these become significant when we compute the Levi-Civita connection and curvature form by differentiation.

space (see [C-Y]). We use Alexeevskii's curvature formula (Theorem 2.3 in this paper) (see [A] and [S]) in an essential way.

Summing up, our main results are the following:

Theorem 1.4. (1) *The Ricci tensor of the Z -metric*

$$g_\lambda^Z = \lambda^2(\hat{\alpha}_1^2 + \hat{\alpha}_3^2) + \sum_{i=0}^3 {}^tX^i \cdot X^i$$

is given by the formula

$$\text{Ric}_\lambda^Z = \frac{4}{\lambda^2} \lambda^2(\hat{\alpha}_1^2 + \hat{\alpha}_3^2) + (4n + 8) \sum_{i=0}^3 {}^tX^i \cdot X^i .$$

In particular ρg_λ^Z is Einstein if and only if $\lambda^2 = \frac{1}{n+2}$.

(2) *The Ricci flow preserves the family $\mathbb{R}_+ \cdot \mathcal{F}^Z$ of Z -metrics. The space $\mathbb{R}_+ \cdot \mathcal{F}^Z$ is foliated by the trajectories of the Ricci flow solutions and these are all ancient solutions. Suppose that the Ricci flow $g(t)$ in $\mathbb{R}_+ \cdot \mathcal{F}^Z$ is defined on $(-\infty, T)$. Then, modulo scaling, the solution $g(t)$ converges to the Einstein metric g_λ^Z as $t \rightarrow -\infty$. Moreover, modulo scaling, the limit $t \rightarrow T$ corresponds to the “collapse” of g_λ^Z where the X^i -direction shrinks faster.*

Applying Bando / Shi estimate to the ancient solutions in Theorem 1.4 (2), we have the limit formula

$$\lim_{\lambda \rightarrow \infty} |\nabla^{g_\lambda^Z} \text{Rm}^{g_\lambda^Z}|_{g_\lambda^Z} = 0 .$$

We will show that this limit formula implies the positive answer to the LeBrun-Salamon conjecture (Conjecture 1.2) claiming that any locally irreducible compact positive quaternion Kähler manifold is isometric to one of the Wolf spaces.

Finally, we remark that the methods used in this paper is a generalization of those used in [K-O] in the case of self-dual positive Einstein 4-manifolds.

§2. Moving Frame Description of Quaternion Kähler Manifolds and their Twistor Spaces. Canonical Deformation Metrics.

Throughout this paper we will use the Einstein summation convention.

We denote the quaternions by \mathbb{H} and identify $\mathbb{R}^{4n} = \mathbb{H}^n$. Then \mathbb{H} operates on $\mathbb{R}^{4n} = \mathbb{H}^n$ from the right which makes $\mathbb{R}^{4n} = \mathbb{H}^n$ a right \mathbb{H} -module. We then define a subgroup $\text{Sp}(n)$ of $\text{SO}(4n)$ as

$$\text{Sp}(n) = \{A \in \text{SO}(4n) \mid A \text{ is } \mathbb{H}\text{-linear}\} .$$

The image in $\text{SO}(4n)$ of the right action of the group $\text{Sp}(1)$ of unit quaternions on \mathbb{H}^n forms a subgroup of $\text{SO}(4n)$ which we denote by $\text{Sp}(1)$ by abuse of notations. Then we define the subgroup $\text{Sp}(n)\text{Sp}(1)$ of $\text{SO}(4n)$ to be the product of the subgroups $\text{Sp}(n)$ and $\text{Sp}(1)$ in $\text{SO}(4n)$. If $n > 1$, the group $\text{Sp}(n)\text{Sp}(1)$ is a proper subgroup of $\text{SO}(4n)$, while if $n = 1$, the group $\text{Sp}(1)\text{Sp}(1)$ coincides with $\text{SO}(4)$. For this reason, we assume $n > 1$ from here on.

Definition 2.1. A $4n$ -dimensional Riemannian manifold is quaternion Kähler if its holonomy group is contained in $Sp(n)Sp(1)$.

A $4n$ -dimensional Riemannian manifold (M^{4n}, g) is quaternion Kähler if and only if the principal $SO(4n)$ -bundle \mathcal{F} of oriented orthonormal frames reduces to an $Sp(n)Sp(1)$ -bundle \mathcal{P} (it is called the *holonomy reduction* of the oriented orthonormal frame bundle). Let $(e_A)_{A=1}^{4n} \in \mathcal{P}$ be an orthonormal frame of M at m . Then we have an identification

$$M_m \rightarrow \mathbb{H}^n, \quad x^A e_A \mapsto (x^a + ix^{n+a} + jx^{2n+a} + kx^{3n+a})_{a=1}^n.$$

Therefore each tangent space M_m becomes a right \mathbb{H} -module. A local section (e^A) of $\mathcal{P} \rightarrow M$ on an open set $U \subset M$ defines a right \mathbb{H} -module structure on TU . However, this does not induce a global right \mathbb{H} -module structure on TM because of the existence of the $Sp(1)$ part in $Sp(n)Sp(1)$.

The right action of i and j on $\mathbb{R}^{4n} = \mathbb{H}^n$ are given by the matrices

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

where 0 (resp. 1) denotes $n \times n$ zero (resp. identity) matrix. Therefore, the Lie algebra $\mathfrak{sp}(n)$ is computed as

$$\begin{pmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{pmatrix}$$

where $A_0 = -{}^t A_0$ and $A_\lambda = {}^t A_\lambda$ ($1 \leq \lambda \leq 4$) are $n \times n$ matrices. Similarly, the Lie algebra of the subgroup $Sp(1)$ of $SO(4n)$ is computed as

$$\begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & a_3 & -a_2 \\ a_2 & -a_3 & 0 & a_1 \\ a_3 & a_2 & -a_1 & 0 \end{pmatrix}$$

where a_1, a_2, a_3 are $n \times n$ scalar matrices. Therefore the Lie algebra $\mathfrak{sp}(n)\mathfrak{sp}(1)$ is expressed as

$$(2-1) \quad \begin{pmatrix} A_0 & -A_1 - a_1 & -A_2 - a_2 & -A_3 - a_3 \\ A_1 + a_1 & A_0 & -A_3 + a_3 & A_2 - a_2 \\ A_2 + a_2 & A_3 - a_3 & A_0 & -A_1 + a_1 \\ A_3 + a_3 & -A_2 + a_2 & A_1 - a_1 & A_0 \end{pmatrix}$$

where $A_0 = -{}^t A_0$ and $A_\lambda = {}^t A_\lambda$ ($1 \leq \lambda \leq 4$) are $n \times n$ matrices and a_1, a_2, a_3 are $n \times n$ scalar matrices. In the following moving frame computation, we will use α_i instead of a_i ($i = 1, 2, 3$) to represent $\mathfrak{sp}(1)$ -valued 1-forms.

Let (M^{4n}, g) be a (locally irreducible) quaternion Kähler manifold, i.e., its holonomy group is subgroup of $SO(4n)$ contained in $Sp(n)Sp(1)$. Let $\mathcal{P} \rightarrow M$ be the holonomy reduction of the bundle of oriented orthonormal frames of M . Then \mathcal{P} is a principal $Sp(n)Sp(1)$ bundle and we say a point of \mathcal{P} (i.e., an orthonormal frame at some point of M) a quaternion orthonormal

frame. The moving frame description of a quaternion Kähler manifold is the following. Each local quaternion orthonormal frame field $e = (e_A)_{A=1}^{4n}$ on an open set $U \subset M$ defines a section $e : U \rightarrow \mathcal{P}$ and therefore its dual coframe field $\theta = (\theta^A)_{A=1}^{4n}$ is pulled back to $e(U) \subset \mathcal{P}$ via the restriction $\pi|_{e(U)} : e(U) \rightarrow U$ of the projection $\mathcal{P} \rightarrow M$ to $e(U)$. We thus get a canonical system of 1-forms $(\theta^A)_{A=1}^{4n}$ on \mathcal{P} such that at each $e = (e_A)_{A=1}^{4n} \in \mathcal{P}$, $(\theta^A)_{A=1}^{4n}$ coincides with its dual coframe pulled back to \mathcal{P} . The Levi-Civita connection ∇ of (M, g) defines a unique right $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -invariant $\mathrm{sp}(n)\mathrm{sp}(1)$ -valued 1-form $\Gamma = (\Gamma^A_B)$ on \mathcal{P} satisfying

$$\nabla_X e_B = e_A \Gamma^A_B(X)$$

for each local quaternion orthonormal frame $(e_A)_{A=1}^{4n}$ and a local vector field X on M . Here, $\Gamma = (\Gamma^A_B)$ can be written as

$$(2-2) \quad (\Gamma^A_B) = \begin{pmatrix} \Gamma_0 & -\Gamma_1 - \alpha_1 & -\Gamma_2 - \alpha_2 & -\Gamma_3 - \alpha_3 \\ \Gamma_1 + \alpha_1 & \Gamma_0 & -\Gamma_3 + \alpha_3 & \Gamma_2 - \alpha_2 \\ \Gamma_2 + \alpha_2 & \Gamma_3 - \alpha_3 & \Gamma_0 & -\Gamma_1 + \alpha_1 \\ \Gamma_3 + \alpha_3 & -\Gamma_2 + \alpha_2 & \Gamma_1 - \alpha_1 & \Gamma_0 \end{pmatrix} =: \Gamma$$

where $\Gamma_0 = -{}^t\Gamma_0$ and $\Gamma_\lambda = {}^t\Gamma_\lambda$ ($1 \leq \lambda \leq 4$) are $n \times n$ matrix valued 1-forms and $\alpha_1, \alpha_2, \alpha_3$ are $n \times n$ scalar matrix valued 1-forms. The 1-forms θ^A and Γ^A_B satisfy the following first and second structure equations:

$$\begin{aligned} d\theta^A + \Gamma^A_B \wedge \theta^B &= 0, \\ d\Gamma^A_B + \Gamma^A_C \wedge \Gamma^C_B &= \Omega^A_B, \end{aligned}$$

where Ω^A_B is the skew symmetric matrix of 2-forms which is identified with the curvature 2-form of (M, g) in the following way. For each point $e = (e_A) \in \mathcal{P}$ on $m \in M$ we have

$$R(X, Y)e_B = e_K \Omega^K_B(X, Y)$$

for all $X, Y \in M_m$. Therefore the sectional curvature for the 2-plane spanned by $\{e_A, e_B\}$ is given by

$$K(e_A, e_B) = g(R(e_A, e_B)e_B, e_A) = g(\Omega^K_B(e_A, e_B)e_K, e_A).$$

We proceed to the description of the twistor space of a quaternion Kähler manifold (M^{4n}, g) . A unit quaternion q is pure imaginary if and only if $q^2 = -1$ holds. Therefore, from the definition of \mathcal{P} , a choice of a quaternion orthonormal frame $(e_A)_{A=1}^{4n} \in \mathcal{P}$ of M_m canonically defines the identification

{unit pure imaginary quaternions}

right action
 \longleftrightarrow

{orthogonal complex structures on M_m } .

This identification itself depends on the basis $(e_A) \in \mathcal{P}$. However, if q is a unit pure imaginary quaternion, then so is $x^{-1}qx$ for any unit quaternion x and therefore the set \mathbb{P}^1 of all orthogonal complex structures on M_m is independent of the choice of the basis $(e_A) \in \mathcal{P}$. The twistor space

\mathcal{Z} of M is by definition the fiber bundle over M consisting of all orthogonal complex structures of all tangent spaces of M . Therefore the twistor space is described as the associated bundle

$$\mathcal{Z} = \mathcal{P} \times_{\mathrm{Sp}(n)\mathrm{Sp}(1)} \mathbb{P}^1$$

where $\mathrm{Sp}(n)\mathrm{Sp}(1)$ operates on the set \mathbb{P}^1 of unit pure imaginary quaternions by the trivial action of $\mathrm{Sp}(n)$ and the canonical right action of the group $\mathrm{Sp}(1)$ of unit quaternions given by $q \mapsto x^{-1}qx$. It follows from the definition of the twistor space that \mathcal{Z} has a canonical almost complex structure. Indeed, the Levi-Civita connection of (M, g) induces the horizontal distribution on the twistor fibration $\mathcal{Z} \rightarrow M$. We then have the canonical complex structure (defined by identifying the set of unit imaginary quaternions with \mathbb{P}^1 by the stereo graphic projection) on each \mathbb{P}^1 -fiber. Moreover, we associate to each horizontal subspace of $H(\mathcal{Z}_z)$ ($z \in \mathcal{Z}$ lies over $m \in M$), the almost complex structure of M_m represented by $z \in \mathcal{Z}$. We have thus defined canonically an almost complex structure on \mathcal{Z} which we call the orthogonal complex structure on \mathcal{Z} . Moreover the Levi-Civita horizontal distribution on the twistor fibration $\mathcal{Z} \rightarrow M$ canonically defines the sum of the scaled fiber Fubini-Study metric and the fixed base metric on M which are expressed as

$$(2-3) \quad g_\lambda^{\mathrm{can}} = \lambda^2 g_{\mathrm{FS}} + g_M .$$

This class of metrics are called the canonical deformation metrics.

Salamon [S] proved the following fundamental result:

Theorem 2.2. (1) *The orthogonal almost complex structure on the twistor space \mathcal{Z} is integrable.*

(2) *Suppose that the scalar curvature of (M, g) is positive. Then there is a unique scaling of the fiber metric such that the canonical deformation metric on the twistor space \mathcal{Z} is Kähler-Einstein with positive scalar curvature.*

We start by recalling the idea of the Cartan formalism of moving frames. Let (N, g) be any n -dimensional oriented Riemannian manifold and $\mathcal{F} \rightarrow N$ the bundle of all oriented orthonormal frames. We have the system $\{\theta^1, \dots, \theta^n\}$ of coframes on \mathcal{F} which is, at $p \in \mathcal{F}$ lying over $m \in N$, the system of 1-forms dual to the orthonormal frame of N_m represented by the point $p \in \mathcal{F}$. Given a local frame field on an open set $U \subset N$, we tautologically associate the section $U \rightarrow \mathcal{F}$. Thus the local frames which are not unique on N becomes a globally defined single valued object on \mathcal{F} and moreover the dual object $\{\theta_1, \dots, \theta_n\}$ consists of differential 1-forms and therefore we have an advantage being able to work functorially on differential forms (such as connection forms) on \mathcal{F} . For instance, the Riemannian metric on N is written as $(\theta^1)^2 + \dots + (\theta^n)^2$ and connection form is computed by taking the exterior differential of $\{\theta_1, \dots, \theta_n\}$ on \mathcal{F} and so on.

Now let us return to our original (quaternion Kähler) situation. A fiber on $m \in M$ of the twistor fibration $\mathcal{Z} \rightarrow M$ is the set of all orthogonal complex structures on the tangent space M_m which is canonically identified with $\mathrm{Sp}(n)\mathrm{Sp}(1)/\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n) \cong \mathbb{P}^1$. Therefore the twistor space is also defined as the orbit space with respect to the $\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$ action on \mathcal{P} , i.e.,

$$\mathcal{Z} = \mathcal{P}/\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n) .$$

We construct local sections $\mathcal{Z} \rightarrow \mathcal{P}$ of the principal $\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$ -bundle $\mathcal{P} \rightarrow \mathcal{Z}$ in the following way (we use these local sections to construct a certain class of metrics on \mathcal{Z}). Fix a point $m \in M$. Let $\mathbb{P}_m^1 \subset \mathcal{Z}$ be the fiber of the twistor fibration over m . To each $z \in \mathbb{P}_m^1$ we (locally) associate a quaternion orthonormal frame in the fiber of $\mathcal{P} \rightarrow M$ over m so that the frame is

ordered in the way compatible with respect to the orthogonal complex structure represented by z . If z varies on \mathbb{P}_m^1 such frames rotates by an element of $\mathrm{Sp}(n)\mathrm{Sp}(1)$ and the rotation is unique modulo those by elements of $\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$. This procedure is possible only locally on \mathbb{P}_m^1 because this is equivalent to make the (local) section of the principal $\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$ -bundle $\mathrm{Sp}(n)\mathrm{Sp}(1) \rightarrow \mathrm{Sp}(n)\mathrm{Sp}(1)/\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n) \cong \mathbb{P}^1$. We extend this construction locally on a small open set $U \subset M$ containing m in such a way that the extended object is parallel at m (one point). We perform this procedure at each point $m \in M$. This way we have constructed local sections of the $\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$ -principal bundle $\mathcal{P} \rightarrow \mathcal{Z}$. We then pull back the canonical 1-forms

$$X^0, X^1, X^2, X^3$$

and the $\mathfrak{sp}(1)$ -part of the connection form

$$\alpha_1, \alpha_2, \alpha_3$$

by the above constructed local sections. We thus get the system of 1-forms

$$\{X^0, X^1, X^2, X^3, \alpha_1, \alpha_3\}$$

locally at 1 point on Z . Then the basic canonical deformation metric in an expression in terms of the orthonormal coframes is expressed as (see [C-Y])

$$g_1^{\mathrm{can}} := (\alpha_1^2 + \alpha_3^2) + {}^tX^0X^0 + {}^tX^1X^1 + {}^tX^2X^2 + {}^tX^3X^3.$$

In the following arguments, we will consider the metrics of the form

$$(2-4) \quad g_\lambda^{\mathrm{can}} := \lambda^2(\alpha_1^2 + \alpha_3^2) + {}^tX^0X^0 + {}^tX^1X^1 + {}^tX^2X^2 + {}^tX^3X^3$$

on \mathcal{Z} (canonical deformation metrics).

In the following discussion, we compute the curvature form of the canonical deformation metrics g_λ^{can} defined on \mathcal{Z} . We use the moving frame computation. The scalings of the various standard metrics are hidden in the computation. To avoid confusion, we fix our scaling convention in the following way :

- We fix the scale of the invariant metric of $\mathbb{H}\mathbb{P}^n$ so that the sectional curvatures range in the interval $[1, 4]$, i.e., $\mathrm{Ric}(g_{\mathbb{H}\mathbb{P}^n}) = 4(n+2)g_{\mathbb{H}\mathbb{P}^n}$ and so $\mathrm{Scal}(g_{\mathbb{H}\mathbb{P}^n}) = 16n(n+2)$. Namely we set

$$\tilde{S} = 16n(n+2)$$

from here on.

- We fix the scale of the Fubini-Study metric of the \mathbb{P}^1 -fiber of the twistor fibration and other cases so that the Gaussian curvature is identically 4.

We will consider the following two types of scalings.

- We fix the $\sum_{i=0}^3 {}^tX^iX^i$ -part and vary the ratio S/\tilde{S} . In this case the above metric turns out to be Kähler (eventually Kähler-Einstein) on \mathcal{Z} if and only if $S/\tilde{S} = 1$. We will use this scaling in the computation in the transversal Kähler situation.

• We normalize the base quaternion Kähler metric g so that $S = \tilde{S}$ holds and scale the $\sum_{i=0}^3 {}^tX^i X^i$ -part by the scaling parameter λ^2 (so that the “curvature becomes λ^{-2} -times the original one in this direction”). In this case the metric g_λ^{can} turns out to be Kähler (eventually Kähler-Einstein) on \mathcal{Z} if and only if $\lambda = 1$.

From here on until the end of §2, we describe the moving frames on the twistor space \mathcal{Z} with the above introduced Z-metrics (and with complex structure in Theorem 2.2 if necessary).

Let $(e_A) \in \mathcal{P}$ be a quaternion orthonormal basis of M_m . This canonically defines an identification M_m with

$$\mathbb{R}^{4n} = \mathbb{H}^n = \mathbb{C}^{2n}$$

by

$$(x^a + ix^{n+a} + jx^{2n+a} + kx^{3n+a})_{a=1}^n \longleftrightarrow (x^a + jx^{2n+a}, x^{n+a} + jx^{3n+a})_{a=1}^n .$$

The multiplication of j from the right on $M_m = \mathbb{C}^{2n}$ induces the canonical almost complex structure J of \mathbb{C}^{2n} . The infinitesimal deformation (“unit velocity vector tangent to a 1-parameter deformation”) of unit imaginary quaternions at j is expressed as $\alpha_1 i + \alpha_3 k$ (this is because the tangent space of the set of unit imaginary quaternions at q is given by the orthogonality condition $\{\sigma \mid \Re(q\sigma) = 0\}$). In this situation we pick a point $z \in \mathcal{Z}$ which induces the almost complex structure on $M_m = \mathbb{C}^{2n}$ corresponding to J . The canonical almost complex structure of \mathcal{Z} at z over m is defined by specifying the basis of all $(1, 0)$ -forms as follows.

$$\begin{aligned} (2-5) \quad \zeta^0 &= \alpha_1 + i\alpha_3 , \\ \zeta^a &= x^a + ix^{2n+a} , \\ \zeta^{n+a} &= x^{n+a} + ix^{3n+a} . \end{aligned}$$

For moving frame computation, we introduce the column vectors X^0, X^1, X^2 and X^3 by setting $X^0 = (x^a)$, $X^1 = (x^{n+a})$, $X^2 = (x^{2n+a})$ and $X^3 = (x^{3n+a})$ where $a = 1, \dots, n$. Then a basis of the \mathbb{C} -vector space of all $(1, 0)$ -forms is simply given by

$$\begin{aligned} \zeta^0 &= \alpha_1 + i\alpha_3 , \\ Z^1 &= X^0 + iX^2 [= (\zeta^a)] , \\ Z^2 &= X^1 + iX^3 [= (\zeta^{n+a})] . \end{aligned}$$

The above argument involving X^j ($j = 0, 1, 2, 3$) and α_i ($i = 1, 3$) is the local explanation of the construction of the section of the $\text{Sp}(n)\text{Sp}(1) \cap \text{U}(2n)$ -principal bundle $\mathcal{P} \rightarrow \mathcal{Z}$ which described above in an abstract way.

We would like to describe the curvature tensor of \mathcal{Z} . To do so, we compute the derivation formula on \mathcal{Z} in terms of this basis of $(1, 0)$ -forms on \mathcal{Z} . Set ${}^tX = ({}^tX^0, {}^tX^1, {}^tX^2, {}^tX^3)$ (${}^tX^i$'s being row vectors). Then the first structure equation

$$dX + \Gamma \wedge X = 0$$

is equivalent to

$$\begin{cases} dZ^1 + \overline{Z}^2 \wedge \zeta^0 + (\Gamma_0 + i(\Gamma_2 + \alpha_2)) \wedge Z^1 + (-\Gamma_1 + i\Gamma_3) \wedge Z^2 = 0 \\ dZ^2 - \overline{Z}^1 \wedge \zeta^0 + (\Gamma_1 + i\Gamma_3) \wedge Z^1 + (\Gamma_0 - i(\Gamma_2 - \alpha_2)) \wedge Z^2 = 0 \end{cases}$$

which in matrix form is expressed as

$$(2-6) \quad d \begin{pmatrix} \zeta^a \\ \zeta^{n+a} \end{pmatrix} = \underbrace{- \begin{pmatrix} \Gamma_0 + i\Gamma_2 & -\Gamma_1 + i\Gamma_3 \\ \Gamma_1 + i\Gamma_3 & \Gamma_0 - i\Gamma_2 \end{pmatrix} \begin{pmatrix} \zeta^a \\ \zeta^{n+a} \end{pmatrix}}_{\text{sp}(n)\text{-action}} - \underbrace{\begin{pmatrix} i\alpha_2 & 0 \\ 0 & i\alpha_2 \end{pmatrix} \begin{pmatrix} \zeta^a \\ \zeta^{n+a} \end{pmatrix} - \begin{pmatrix} 0 & -(\alpha_1 + i\alpha_3) \\ \alpha_1 + i\alpha_3 & 0 \end{pmatrix} \begin{pmatrix} \bar{\zeta}^a \\ \bar{\zeta}^{n+a} \end{pmatrix}}_{\text{sp}(1)\text{-action}}$$

What we must do next is to compute $d\zeta^0$. We need the second structure equation

$$d\Gamma + \Gamma \wedge \Gamma = \Omega .$$

The expression (2-2) for the connection form implies the following:

$$\Omega = \begin{pmatrix} \Omega_0^0 & \Omega_1^0 & \Omega_2^0 & \Omega_3^0 \\ \Omega_0^1 & \Omega_0^0 & \Omega_2^1 & \Omega_3^1 \\ \Omega_0^2 & \Omega_1^2 & \Omega_0^0 & \Omega_3^2 \\ \Omega_0^3 & \Omega_1^3 & \Omega_2^3 & \Omega_0^0 \end{pmatrix}$$

where ${}^t\Omega_0^0 = -\Omega_0^0$, $\Omega_\nu^\mu = {}^t\Omega_\nu^\mu = -\Omega_\mu^\nu$. Let (μ, η, ν) be any cyclic permutation of $(1, 2, 3)$. Then we have

$$(2-7) \quad \begin{aligned} \Omega_0^0 &= d\Gamma_0 + \Gamma_0 \wedge \Gamma_0 - \Gamma_1 \wedge \Gamma_1 - \Gamma_2 \wedge \Gamma_2 - \Gamma_3 \wedge \Gamma_3 , \\ \Omega_0^\mu &= d(\Gamma_\mu + \alpha_\mu) + (\Gamma_\mu + \alpha_\mu) \wedge \Gamma_0 + (\Gamma_\eta - \alpha_\eta) \wedge (\Gamma_\nu - \alpha_\nu) \\ &\quad + \Gamma_0 \wedge (\Gamma_\mu + \alpha_\mu) + (-\Gamma_\nu + \alpha_\nu) \wedge (\Gamma_\eta + \alpha_\eta) \\ &= \underbrace{d\Gamma_\mu + \Gamma_\mu \wedge \Gamma_0 + \Gamma_\eta \wedge \Gamma_\nu + \Gamma_0 \wedge \Gamma_\mu - \Gamma_\nu \wedge \Gamma_\eta}_{\Gamma\text{-part}} \\ &\quad + \underbrace{d\alpha_\mu - 2\alpha_\eta \wedge \alpha_\nu}_{\text{scalar part}} , \\ \Omega_\nu^\eta &= d(-\Gamma_\mu + \alpha_\mu) + (\Gamma_\eta + \alpha_\eta) \wedge (-\Gamma_\nu - \alpha_\nu) + (-\Gamma_\mu + \alpha_\mu) \wedge \Gamma_0 \\ &\quad + (\Gamma_\nu - \alpha_\nu) \wedge (\Gamma_\eta - \alpha_\eta) + \Gamma_0 \wedge (-\Gamma_\mu + \alpha_\mu) \\ &= \underbrace{-d\Gamma_\mu - \Gamma_\eta \wedge \Gamma_\nu - \Gamma_\mu \wedge \Gamma_0 + \Gamma_\nu \wedge \Gamma_\eta - \Gamma_0 \wedge \Gamma_\mu}_{\Gamma\text{-part}} \\ &\quad + \underbrace{d\alpha_\mu - 2\alpha_\eta \wedge \alpha_\nu}_{\text{scalar part}} . \end{aligned}$$

Therefore, to compute $d\zeta^0$, we need to know the structure of the curvature tensor of a quaternion Kähler manifold. In fact, the curvature tensor of a quaternion Kähler manifold is very special as is described in the following Alekseevskii's decomposition theorem (see [A] and [S]):

Theorem 2.3. (1) *A locally irreducible quaternion Kähler manifold is Einstein.*

(2) *The curvature operator of a locally irreducible quaternion Kähler manifold (M^{4n}, g) is of the form*

$$\Omega = (S/\tilde{S})\tilde{\Omega} + \Omega'$$

where $\tilde{\Omega}$ is the curvature operator of $\mathbb{H}\mathbb{P}^n$, \tilde{S} is the scalar curvature of $\mathbb{H}\mathbb{P}^n$, S is the scalar curvature of M (S is a constant), and

$$\Omega' \in \text{Sym}^2(\mathfrak{sp}(n)) \subset \text{Sym}^2(\Lambda^2 T^*M) .$$

As was declared in §0, we restrict our attention to locally irreducible quaternion Kähler manifolds and we say just “quaternion Kähler” instead of “locally irreducible quaternion Kähler”.

The meaning of Theorem 2.3 is the following. If we compute the curvature operator of a quaternion Kähler manifold (M^{4n}, g) in terms of the quaternion orthonormal basis (X^A) , then the curvature operator decomposes into the scalar multiple of $\tilde{\Omega}$ and the remaining part. The first part is $(S/\tilde{S})\tilde{\Omega}$ where $\tilde{\Omega}$ is expressed exactly in the same form as the curvature operator of $\mathbb{H}\mathbb{P}^n$ where (X^A) is regarded as quaternion orthonormal for the canonical metric of $\mathbb{H}\mathbb{P}^n$. The remaining part Ω' then looks like a curvature operator of a hyper-Kähler manifold (in particular no scalar α -part is involved).

The $\text{Sp}(n)\text{Sp}(1)$ -principal bundle \mathcal{P} of quaternion orthonormal frames of $\mathbb{H}\mathbb{P}^n$ coincides with the group $\text{Sp}(n+1)/\mathbb{Z}_2$ and therefore the curvature tensor of $\mathbb{H}\mathbb{P}^n$ is computed from the Maurer-Cartan equation on $\text{Sp}(n+1)$ applied to the description of the quaternion projective space

$$\mathbb{H}\mathbb{P}^n = \frac{\text{Sp}(n+1)}{\text{Sp}(n) \times \text{Sp}(1)} = \frac{\text{Sp}(n+1)/\mathbb{Z}_2}{\text{Sp}(n)\text{Sp}(1)}$$

as a symmetric space. The subgroup $\text{Sp}(n)\text{Sp}(1)$ of $\text{Sp}(n+1)$ defined by the Lie algebra embedding

$$\begin{pmatrix} A_0 & -A_1 - a_1 & -A_2 - a_2 & -A_3 - a_3 \\ A_1 + a_1 & A_0 & -A_3 + a_3 & A_2 - a_2 \\ A_2 + a_2 & A_3 - a_3 & A_0 & -A_1 + a_1 \\ A_3 + a_3 & -A_2 + a_2 & A_1 - a_1 & A_0 \end{pmatrix} \mapsto \\ \begin{pmatrix} 0 & a_1 & -a_3 & a_2 \\ -a_1 & 0 & a_2 & a_3 \\ a_3 & -a_2 & 0 & a_1 \\ -a_2 & -a_3 & -a_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} A_0 & -A_1 & A_3 & -A_2 \\ A_1 & A_0 & -A_2 & -A_3 \\ -A_3 & A_2 & A_0 & -A_1 \\ A_2 & A_3 & A_1 & A_0 \end{pmatrix}$$

where a_μ 's are regarded as scalar $n \times n$ matrices in the LHS while in the RHS these are regarded just as scalars. Indeed, the Lie algebra

$$\begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & a_3 & -a_2 \\ a_2 & -a_3 & 0 & a_1 \\ a_3 & a_2 & -a_1 & 0 \end{pmatrix} \leftrightarrow a_1 R_i + a_2 R_j + a_3 R_k$$

of $\mathfrak{sp}(1)$ considered as a Lie subalgebra of $\mathfrak{so}(4n)$ stems from the matrix expression of the right action of quaternions on \mathbb{H}^n . Let R_q (resp. L_q) denote the right (resp. left) action of a quaternion q on \mathbb{H}^n . The correspondence

$$R_i \mapsto -L_i, \quad R_j \mapsto -L_k, \quad R_k \mapsto L_j$$

defines a Lie algebra isomorphism of $\mathfrak{sp}(1)$ which converts the right action of \mathbb{H} on \mathbb{H}^n to the left action. This Lie algebra isomorphism converts the above expression to

$$\begin{pmatrix} 0 & a_1 & -a_3 & a_2 \\ -a_1 & 0 & a_2 & a_3 \\ a_3 & -a_2 & 0 & a_1 \\ -a_2 & -a_3 & -a_1 & 0 \end{pmatrix} \leftrightarrow a_1(-L_i) + a_3L_j + a_2(-L_k) .$$

To obtain a general expression of the Lie algebra $\mathfrak{sp}(n+1)$ we put

$$\begin{pmatrix} 0 & a_1 & -a_3 & a_2 & -{}^tX^0 & -{}^tX^1 & {}^tX^3 & -{}^tX^2 \\ -a_1 & 0 & a_2 & a_3 & -{}^tX^{01} & -{}^tX^{11} & -{}^tX^{21} & -{}^tX^{31} \\ a_3 & -a_2 & 0 & a_1 & -{}^tX^{02} & -{}^tX^{12} & -{}^tX^{22} & -{}^tX^{32} \\ -a_2 & -a_3 & -a_1 & 0 & -{}^tX^{03} & -{}^tX^{13} & -{}^tX^{23} & -{}^tX^{33} \\ X^0 & X^{01} & X^{02} & X^{03} & A_0 & -A_1 & A_3 & -A_2 \\ X^1 & X^{11} & X^{12} & X^{13} & A_1 & A_0 & -A_2 & -A_3 \\ -X^3 & X^{21} & X^{22} & X^{23} & -A_3 & A_2 & A_0 & -A_1 \\ X^2 & X^{31} & X^{32} & X^{33} & A_2 & A_3 & A_1 & A_0 \end{pmatrix}$$

and determine $X^{\mu\nu}$'s from the commutativity with the right action of i and j , namely, with the matrices

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -E_n & 0 & 0 \\ E_n & 0 & 0 & 0 \\ 0 & 0 & 0 & E_n \\ 0 & 0 & -E_n & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & -E_n & 0 \\ 0 & 0 & 0 & -E_n \\ E_n & 0 & 0 & 0 \\ 0 & E_n & 0 & 0 \end{pmatrix} .$$

It follows that the Lie algebra $\mathfrak{sp}(n+1)$ is expressed as

$$\begin{pmatrix} 0 & a_1 & -a_3 & a_2 & -{}^tX^0 & -{}^tX^1 & {}^tX^3 & -{}^tX^2 \\ -a_1 & 0 & a_2 & a_3 & {}^tX^1 & -{}^tX^0 & -{}^tX^2 & -{}^tX^3 \\ a_3 & -a_2 & 0 & a_1 & -{}^tX^3 & {}^tX^2 & -{}^tX^0 & -{}^tX^1 \\ -a_2 & -a_3 & -a_1 & 0 & {}^tX^2 & {}^tX^3 & {}^tX^1 & -{}^tX^0 \\ X^0 & -X^1 & X^3 & -X^2 & A_0 & -A_1 & A_3 & -A_2 \\ X^1 & X^0 & -X^2 & -X^3 & A_1 & A_0 & -A_2 & -A_3 \\ -X^3 & X^2 & X^0 & -X^1 & -A_3 & A_2 & A_0 & -A_1 \\ X^2 & X^3 & X^1 & X^0 & A_2 & A_3 & A_1 & A_0 \end{pmatrix} .$$

Therefore, the following matrix represents a basis of left invariant 1-forms on $\mathrm{Sp}(n+1)$:

$$\tilde{\Gamma}_{\alpha,X} := \begin{pmatrix} 0 & \tilde{\alpha}_1 & -\tilde{\alpha}_3 & \tilde{\alpha}_2 & -{}^tX^0 & -{}^tX^1 & {}^tX^3 & -{}^tX^2 \\ -\tilde{\alpha}_1 & 0 & \tilde{\alpha}_2 & \tilde{\alpha}_3 & {}^tX^1 & -{}^tX^0 & -{}^tX^2 & -{}^tX^3 \\ \tilde{\alpha}_3 & -\tilde{\alpha}_2 & 0 & \tilde{\alpha}_1 & -{}^tX^3 & {}^tX^2 & -{}^tX^0 & -{}^tX^1 \\ -\tilde{\alpha}_2 & -\tilde{\alpha}_3 & -\tilde{\alpha}_1 & 0 & {}^tX^2 & {}^tX^3 & {}^tX^1 & -{}^tX^0 \\ X^0 & -X^1 & X^3 & -X^2 & \tilde{\Gamma}_0 & -\tilde{\Gamma}_1 & \tilde{\Gamma}_3 & -\tilde{\Gamma}_2 \\ X^1 & X^0 & -X^2 & -X^3 & \tilde{\Gamma}_1 & \tilde{\Gamma}_0 & -\tilde{\Gamma}_2 & -\tilde{\Gamma}_3 \\ -X^3 & X^2 & X^0 & -X^1 & -\tilde{\Gamma}_3 & \tilde{\Gamma}_2 & \tilde{\Gamma}_0 & -\tilde{\Gamma}_1 \\ X^2 & X^3 & X^1 & X^0 & \tilde{\Gamma}_2 & \tilde{\Gamma}_3 & \tilde{\Gamma}_1 & \tilde{\Gamma}_0 \end{pmatrix} .$$

The Maurer-Cartan equation

$$d\tilde{\Gamma}_{\alpha,X} + \tilde{\Gamma}_{\alpha,X} \wedge \tilde{\Gamma}_{\alpha,X} = 0$$

implies the following. Let (μ, η, ν) be any cyclic permutation of $(1, 2, 3)$. Then :

$$\begin{aligned} & d\tilde{\Gamma}_0 - X^0 \wedge {}^tX^0 - X^1 \wedge {}^tX^1 - X^2 \wedge {}^tX^2 - X^3 \wedge {}^tX^3 \\ & + \tilde{\Gamma}_0 \wedge \tilde{\Gamma}_0 - \tilde{\Gamma}_1 \wedge \tilde{\Gamma}_1 - \tilde{\Gamma}_2 \wedge \tilde{\Gamma}_2 - \tilde{\Gamma}_3 \wedge \tilde{\Gamma}_3 = 0 , \\ & d\tilde{\alpha}_\mu - 2\tilde{\alpha}_\eta \wedge \tilde{\alpha}_\nu - {}^tX^\mu \wedge X^0 + {}^tX^0 \wedge X^\mu + {}^tX^\nu \wedge X^\eta - {}^tX^\eta \wedge X^\nu = 0 , \\ & d\tilde{\Gamma}_\mu + \tilde{\Gamma}_\mu \wedge \tilde{\Gamma}_0 + \tilde{\Gamma}_0 \wedge \tilde{\Gamma}_\mu - \tilde{\Gamma}_\nu \wedge \tilde{\Gamma}_\eta + \tilde{\Gamma}_\eta \wedge \tilde{\Gamma}_\nu \\ & - X^\mu \wedge {}^tX^0 + X^0 \wedge {}^tX^\mu - X^\nu \wedge {}^tX^\eta + X^\eta \wedge {}^tX^\nu = 0 . \end{aligned}$$

This and (2-4) implies that the curvature $(\tilde{\Omega}_\nu^\mu)$ of $\mathbb{H}\mathbb{P}^n$ is expressed as

$$\begin{aligned} (2-8) \quad & \tilde{\Omega}_0^0 = d\tilde{\Gamma}_0 - \tilde{\Gamma}_1 \wedge \tilde{\Gamma}_1 - \tilde{\Gamma}_2 \wedge \tilde{\Gamma}_2 - \tilde{\Gamma}_3 \wedge \tilde{\Gamma}_3 \\ & = X^0 \wedge {}^tX^0 + X^1 \wedge {}^tX^1 + X^2 \wedge {}^tX^2 + X^3 \wedge {}^tX^3 , \\ & \tilde{\Omega}_0^\mu = d\tilde{\Gamma}_\mu + \tilde{\Gamma}_\mu \wedge \tilde{\Gamma}_0 + \tilde{\Gamma}_0 \wedge \tilde{\Gamma}_\mu - \tilde{\Gamma}_\nu \wedge \tilde{\Gamma}_\eta + \tilde{\Gamma}_\eta \wedge \tilde{\Gamma}_\nu \\ & \quad + d\tilde{\alpha}_\mu - 2\tilde{\alpha}_\eta \wedge \tilde{\alpha}_\nu \\ & = X^\mu \wedge {}^tX^0 - X^0 \wedge {}^tX^\mu + X^\nu \wedge {}^tX^\eta - X^\eta \wedge {}^tX^\nu \\ & \quad + 2({}^tX^\mu \wedge X^0 + {}^tX^\eta \wedge X^\nu) , \\ & \tilde{\Omega}_\nu^\eta = -d\tilde{\Gamma}_\mu - \tilde{\Gamma}_\eta \wedge \tilde{\Gamma}_\nu + \tilde{\Gamma}_\nu \wedge \tilde{\Gamma}_\eta - \tilde{\Gamma}_0 \wedge \tilde{\Gamma}_\mu - \tilde{\Gamma}_\mu \wedge \tilde{\Gamma}_0 \\ & \quad + d\tilde{\alpha}_\mu - 2\tilde{\alpha}_\eta \wedge \tilde{\alpha}_\nu \\ & = -X^\mu \wedge {}^tX^0 + X^0 \wedge {}^tX^\mu - X^\nu \wedge {}^tX^\eta + X^\eta \wedge {}^tX^\nu \\ & \quad + 2({}^tX^\mu \wedge X^0 + {}^tX^\eta \wedge X^\nu) . \end{aligned}$$

In particular this implies that the sectional curvatures of $\mathbb{H}\mathbb{P}^n$ are, for instance, $K(e_1, e_a) = K(e_1, e_{n+a}) = K(e_1, e_{2n+a}) = K(e_1, e_{3n+a}) = 1$ ($2 \leq a \leq n$) and $K(e_1, e_{n+1}) = K(e_1, e_{2n+1}) = K(e_1, e_{3n+1}) = 4$ (i.e., $\mathbb{H}\mathbb{P}^n$ is 1/4-pinned). This implies that $\mathbb{H}\mathbb{P}^n$ is Einstein with $\text{Ric}(g) = 4(n+2)g$.

We now return to the computation of the structure equation with respect to the class of metrics on the twistor space \mathcal{Z} introduced above for general quaternion Kähler manifolds.

We give a proof of Theorem 2.2 by following the arguments in [C-Y], since the proof of our main theorem is based on the moving frame proof of Theorem 2.2 in the technical level. Indeed, what is essential in the proof of our main theorem is the comparison of the canonical deformation and Z-metrics (which we introduce in §3) on the twistor space \mathcal{Z} in terms of the Cartan formalism of the moving frames.

We start with the case of $M = \mathbb{H}\mathbb{P}^n$. We compute in the transversal Kähler setting. To compute $d\zeta^0$ ($\zeta^0 = \alpha_1 + i\alpha_3$), we use the formula

$$(2-9) \quad d\tilde{\alpha}_\mu - 2\tilde{\alpha}_\eta \wedge \tilde{\alpha}_\nu = 2({}^tX^\mu \wedge X^0 + {}^tX^\eta \wedge X^\nu) .$$

This implies

$$d\tilde{\zeta}^0 = -2i\tilde{\alpha}_2 \wedge \tilde{\zeta}^0 + {}^tZ^2 \wedge Z^1 - {}^tZ^1 \wedge Z^2 .$$

It follows from (2-3) and (2-6) the structure equation on the twistor space of $\mathbb{H}\mathbb{P}^n$:

$$(2-10) \quad d \begin{pmatrix} \tilde{\zeta}^0 \\ Z^1 \\ Z^2 \end{pmatrix} = - \begin{pmatrix} 2i\tilde{\alpha}_2 & -{}^t Z^2 & {}^t Z^1 \\ \overline{Z}^2 & \tilde{\Gamma}_0 + i\tilde{\Gamma}_2 + i\tilde{\alpha}_2 & -\tilde{\Gamma}_1 + i\tilde{\Gamma}_3 \\ -\overline{Z}^1 & \tilde{\Gamma}_1 + i\tilde{\Gamma}_3 & \tilde{\Gamma}_0 - i\tilde{\Gamma}_2 + i\tilde{\alpha}_2 \end{pmatrix} \wedge \begin{pmatrix} \tilde{\zeta}^0 \\ Z^1 \\ Z^2 \end{pmatrix}.$$

We then turn to the general case, i.e., M being a quaternion Kähler manifold, \mathcal{Z} (resp. $\tilde{\mathcal{Z}}$) its twistor space (resp. extended twistor space). We first consider the transversal Kähler setting.

We recall that ζ^0 , Z^1 and Z^2 is a basis of $(1,0)$ -forms at a point of \mathcal{Z} w.r.to the canonical complex structure. It follows from (2-3) that

$$d \begin{pmatrix} \zeta^0 \\ Z^1 \\ Z^2 \end{pmatrix} = - \begin{pmatrix} 2i\alpha_2 & * & * \\ \overline{Z}^2 & \Gamma_0 + i\Gamma_2 + i\alpha_2 & -\Gamma_1 + i\Gamma_3 \\ -\overline{Z}^1 & \Gamma_1 + i\Gamma_3 & \Gamma_0 - i\Gamma_2 + i\alpha_2 \end{pmatrix} \wedge \begin{pmatrix} \zeta^0 \\ Z^1 \\ Z^2 \end{pmatrix} + \dots$$

and therefore what we have to compute is to express $d\alpha_\mu - 2\alpha_\eta \wedge \alpha_\nu$ in terms of Z^μ 's (ζ^0 being $\alpha_1 + i\alpha_3$). To compute $d\zeta^0$, we observe from (2-4) that

$$(2-11) \quad \Omega_0^\mu + \Omega_\nu^\eta = 2d\alpha_\mu - 4\alpha_\eta \wedge \alpha_\nu.$$

holds (μ, η, ν being any cyclic permutation of $(1, 2, 3)$). Combining (2-8) with Theorem 2.3, we get

$$(2-12) \quad \begin{aligned} d\alpha_\mu - 2\alpha_\eta \wedge \alpha_\nu &= \frac{1}{2}(\Omega_0^\mu + \Omega_\nu^\eta) \\ &= \frac{1}{2}(S/\tilde{S})(\tilde{\Omega}_0^\mu + \tilde{\Omega}_\nu^\eta) + \frac{1}{2}(\Omega'^\mu + \Omega'^\eta) \\ &= (S/\tilde{S})(d\tilde{\alpha}_\mu - 2\tilde{\alpha}_\eta \wedge \tilde{\alpha}_\nu) \quad [\text{because } \Omega' \text{ part does not involve the } \alpha\text{-part}] \\ &= 2(S/\tilde{S})({}^t X^\mu \wedge X^0 + {}^t X^\eta \wedge X^\nu). \end{aligned}$$

The formula (2-12) implies that the $\text{SO}(2)$ -bundle $\tilde{\mathcal{Z}} \rightarrow \mathcal{Z}$ defines a Hermitian holomorphic line bundle on the twistor space \mathcal{Z} with its curvature form proportional to the Kähler-Einstein metric of \mathcal{Z} . We get from formula (2-12) the formula

$$(2-13) \quad \begin{aligned} d\zeta^0 &= d(\alpha_1 + i\alpha_3) \\ &= 2\alpha_2 \wedge \alpha_3 + (d\alpha_1 - 2\alpha_2 \wedge \alpha_3) + 2i\alpha_1 \wedge \alpha_2 + i(d\alpha_3 - 2\alpha_1 \wedge \alpha_2) \\ &= -2i\alpha_2 \wedge \zeta^0 + (S/\tilde{S})({}^t Z^2 \wedge Z^1 - {}^t Z^1 \wedge Z^2) \end{aligned}$$

which means that the twistor space \mathcal{Z} is a *holomorphic contact manifold*. It follows that

$$(2-13') \quad d \begin{pmatrix} \zeta^0 \\ Z^1 \\ Z^2 \end{pmatrix} = - \begin{pmatrix} 2i\alpha_2 & -(S/\tilde{S}){}^t Z^2 & (S/\tilde{S}){}^t Z^1 \\ \overline{Z}^2 & \Gamma_0 + i\Gamma_2 + i\alpha_2 & -\Gamma_1 + i\Gamma_3 \\ -\overline{Z}^1 & \Gamma_1 + i\Gamma_3 & \Gamma_0 - i\Gamma_2 + i\alpha_2 \end{pmatrix} \wedge \begin{pmatrix} \zeta^0 \\ Z^1 \\ Z^2 \end{pmatrix}.$$

Let Γ denote the matrix in the right hand side of (2-9). The formula (2-9) confirms that the almost complex structure defined by the basis $\{\zeta^0, Z^1, Z^2\}$ on \mathcal{Z} of the space of $(1,0)$ -forms is integrable,

since the right hand side contains no $(0, 2)$ -forms. Moreover, the matrix in (2-13') becomes skew-Hermitian if and only if the metric of M is scaled so that $S/\tilde{S} = 1$. This means that the Hermitian metric of \mathcal{Z} with the property that the basis $\{\zeta^0, Z^1, Z^2\}$ of $(1, 0)$ -forms is unitary, is a Kähler metric of \mathcal{Z} if and only if the metric of M is scaled so that $S/\tilde{S} = 1$. Now let the metric M is scaled so that $S/\tilde{S} = 1$. The second structure equation $d\Gamma + \Gamma \wedge \Gamma = \Omega$ computes the curvature Ω of the Kähler metric $\zeta^0 \wedge \bar{\zeta}^0 + {}^t Z^1 \wedge \bar{Z}^1 + {}^t Z^2 \wedge \bar{Z}^2$. The direct computation shows the following :

$$\begin{pmatrix} 2\zeta^0 \wedge \bar{\zeta}^0 + {}^t Z^1 \wedge \bar{Z}^1 & \zeta^0 \wedge {}^t \bar{Z}^1 & \zeta^0 \wedge {}^t \bar{Z}^2 \\ + {}^t Z^2 \wedge \bar{Z}^2 & & \\ Z^1 \wedge \bar{\zeta}^0 & \Omega_0^0 + i\Omega_0^2 & -\frac{1}{2}\{\Omega_0^1 + \Omega_2^3 - i(\Omega_0^3 + \Omega_1^2)\} \\ -\bar{Z}^2 \wedge {}^t Z^2 + \zeta^0 \wedge \bar{\zeta}^0 & & + \bar{Z}^2 \wedge {}^t Z^1 \\ Z^2 \wedge \bar{\zeta}^0 & \frac{1}{2}\{\Omega_0^1 + \Omega_2^3 + i(\Omega_0^3 + \Omega_1^2)\} & \Omega_0^0 + i\Omega_1^3 \\ + \bar{Z}^1 \wedge {}^t Z^2 & & -\bar{Z}^1 \wedge {}^t Z^1 + \zeta^0 \wedge \bar{\zeta}^0 \end{pmatrix}$$

which is certainly skew-Hermitian. The Ricci form of the Kähler metric $\zeta^0 \wedge \bar{\zeta}^0 + {}^t Z^1 \wedge \bar{Z}^1 + {}^t Z^2 \wedge \bar{Z}^2$ (identified with g_1^{can}) is given by

$$\text{Ric}(\Omega) = \text{tr}(\Omega) = 2(n+1) \left\{ \zeta^0 \wedge \bar{\zeta}^0 + {}^t Z^1 \wedge \bar{Z}^1 + {}^t Z^2 \wedge \bar{Z}^2 \right\}.$$

This implies that the Kähler metric $\zeta^0 \wedge \bar{\zeta}^0 + {}^t Z^1 \wedge \bar{Z}^1 + {}^t Z^2 \wedge \bar{Z}^2$ (this is g_1^{can}) is Kähler-Einstein. This complete the proof of Theorem 2.2 \square

We are now ready to compute the Levi-Civita connection of the canonical deformation metrics. We have, at $z \in \mathcal{Z}$ modulo terms vanishing to order (≥ 2) , the following matrix expression :

$$(2-14) \quad d \begin{pmatrix} \lambda\alpha_1 \\ \lambda\alpha_3 \\ X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} + \begin{pmatrix} 0 & -2\alpha_2 & -\lambda^t X^1 & \lambda^t X^0 & \lambda^t X^3 & -\lambda^t X^2 \\ 2\alpha_2 & 0 & -\lambda^t X^3 & \lambda^t X^2 & -\lambda^t X^1 & \lambda^t X^0 \\ \lambda X^1 & \lambda X^3 & \Gamma_0 & -\Gamma_1^- & -\Gamma_2^+ & -\Gamma_3^- \\ -\lambda X^0 & -\lambda X^2 & \Gamma_1^- & \Gamma_0 & -\Gamma_3^+ & \Gamma_2^- \\ -\lambda X^3 & \lambda X^1 & \Gamma_2^+ & \Gamma_3^+ & \Gamma_0 & -\Gamma_1^+ \\ \lambda X^2 & -\lambda X^0 & \Gamma_3^- & -\Gamma_2^- & \Gamma_1^+ & \Gamma_0 \end{pmatrix} \wedge \begin{pmatrix} \lambda\alpha_1 \\ \lambda\alpha_3 \\ X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $\Gamma_1^\pm = \Gamma_1 \pm (\lambda^2 - 1)\alpha_1$, $\Gamma_3^\pm = \Gamma_3 \pm (\lambda^2 - 1)\alpha_3$ and $\Gamma_2^\pm = \Gamma_2 \pm (\lambda^2 - 1)\alpha_2$. The skew symmetric matrix in (2-14) is the the connection form of the Levi-Civita connection which we denote as Γ_λ . (2-14) is the first structure equation for the canonical deformation metric g_λ^{can} on \mathcal{Z} and the matrix part should be understood as 1-form germs at $z \in \mathcal{Z}$ modulo terms vanishing at z to the order ≥ 2 . Putting $\zeta^0 = \alpha_1 + i\alpha_3$, $Z^1 = X^0 + iX^2$ and $Z^2 = X^1 + iX^3$, (2-14) is rewritten as

$$(2-14') \quad d \begin{pmatrix} \lambda\zeta^0 \\ Z^1 \\ Z^2 \end{pmatrix} + \begin{pmatrix} 2i\alpha_2 & -\lambda^t Z^2 & \lambda^t Z^1 \\ (\lambda - \lambda + \lambda^{-1})\bar{Z}^2 & \Gamma_0 + i\Gamma_2 + i\alpha_2 & -\Gamma_1 + i\Gamma_3 \\ -(\lambda + \lambda - \lambda^{-1})\bar{Z}^1 & \Gamma_1 + i\Gamma_3 & \Gamma_0 - i\Gamma_2 + i\alpha_2 \end{pmatrix} \begin{pmatrix} \lambda\zeta^0 \\ Z^1 \\ Z^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The matrix in (2-14) is skew Hermitian if and only if $\lambda^2 = 1$, i.e., the metric g_λ^{can} is the basic canonical deformation metric g_1^{can} (this reproves Theorem 2.2).

The second structure equation $d\Gamma_\lambda + \Gamma_\lambda \wedge \Gamma_\lambda = \Omega_\lambda$ computes the curvature form $\Omega_\lambda^{\text{can}}$. Put

$$\Omega_\lambda = \begin{pmatrix} \Omega_{\lambda-2}^{-2} & \Omega_{\lambda-1}^{-2} & \Omega_{\lambda 0}^{-2} & \Omega_{\lambda 1}^{-2} & \Omega_{\lambda 2}^{-2} & \Omega_{\lambda 3}^{-2} \\ \Omega_{\lambda-2}^{-1} & \Omega_{\lambda-1}^{-1} & \Omega_{\lambda 0}^{-1} & \Omega_{\lambda 1}^{-1} & \Omega_{\lambda 2}^{-1} & \Omega_{\lambda 3}^{-1} \\ \Omega_{\lambda-2}^0 & \Omega_{\lambda-1}^0 & \Omega_{\lambda 0}^0 & \Omega_{\lambda 1}^0 & \Omega_{\lambda 2}^0 & \Omega_{\lambda 3}^0 \\ \Omega_{\lambda-2}^1 & \Omega_{\lambda-1}^1 & \Omega_{\lambda 0}^1 & \Omega_{\lambda 1}^1 & \Omega_{\lambda 2}^1 & \Omega_{\lambda 3}^1 \\ \Omega_{\lambda-2}^2 & \Omega_{\lambda-1}^2 & \Omega_{\lambda 0}^2 & \Omega_{\lambda 1}^2 & \Omega_{\lambda 2}^2 & \Omega_{\lambda 3}^2 \\ \Omega_{\lambda-2}^3 & \Omega_{\lambda-1}^3 & \Omega_{\lambda 0}^3 & \Omega_{\lambda 1}^3 & \Omega_{\lambda 2}^3 & \Omega_{\lambda 3}^3 \end{pmatrix}$$

Then we have

$$\begin{aligned} \Omega_{\lambda-2}^{-2} &= 0, \quad \Omega_{\lambda-1}^{-1} = 0, \\ \Omega_{\lambda-2}^{-1} &= 2d\alpha_2 - \lambda^2 {}^tX^3 \wedge X^1 - \lambda^2 {}^tX^2 \wedge X^0 + \lambda^2 {}^tX^1 \wedge X^3 + \lambda^2 {}^tX^0 \wedge X^2 \\ &= 4\alpha_3 \wedge \alpha_1 + (4 - 2\lambda^2) {}^tX^3 \wedge X^1 + (4 - 2\lambda^2) {}^tX^2 \wedge X^0 \end{aligned}$$

In the rest of the curvature computation we use (2-2) and the first structure equation $dX + \Gamma \wedge X = 0$. A typical computation is

$$\begin{aligned} \Omega_{\lambda-2}^0 &= \lambda dX^1 + \lambda X^3 \wedge (2\alpha_2) + \Gamma_0 \wedge (\lambda X^1) + (\Gamma_1 + (1 - \lambda^2)\alpha_1) \wedge (\lambda X^0) \\ &\quad + (\Gamma_2 + \alpha_2) \wedge (\lambda X^3) - (\Gamma_3 + (1 - \lambda^2)\alpha_3) \wedge (\lambda X^2) \} \\ &= \lambda \{ -(\Gamma_1 + \alpha_1) \wedge X^0 - \Gamma_0 \wedge X^1 + (\Gamma_3 - \alpha_3) \wedge X^2 - (\Gamma_2 - \alpha_2) \wedge X^3 \} \\ &\quad - 2\lambda\alpha_2 \wedge X^3 + \lambda\Gamma_0 \wedge X^1 + \Gamma_1 \wedge X^0 + \lambda(1 - \lambda^2)\alpha_1 \wedge X^0 \\ &\quad + \lambda\Gamma_2 \wedge X^3 + \lambda\alpha_2 \wedge X^3 - \lambda\Gamma_3 \wedge X^2 - \lambda(1 - \lambda^2)\alpha_3 \wedge X^2 \\ &= \lambda^3 X^0 \wedge \alpha_1 + (2\lambda - \lambda^3) X^2 \wedge \alpha_3. \end{aligned}$$

By similar computations as above, we have the following expression of a part of the curvature :

$$\begin{aligned} \Omega_{\lambda-2}^0 &= \lambda^3 X^0 \wedge \alpha_1 + (2\lambda - \lambda^3) X^2 \wedge \alpha_3, & \Omega_{\lambda-1}^0 &= \lambda^3 X^0 \wedge \alpha_3 - (2\lambda - \lambda^3) X^2 \wedge \alpha_1, \\ \Omega_{\lambda-2}^1 &= \lambda^3 X^1 \wedge \alpha_1 + (2\lambda - \lambda^3) X^3 \wedge \alpha_3, & \Omega_{\lambda-1}^1 &= \lambda^3 X^1 \wedge \alpha_3 - (2\lambda - \lambda^3) X^3 \wedge \alpha_1, \\ \Omega_{\lambda-2}^2 &= \lambda^3 X^2 \wedge \alpha_1 - (2\lambda - \lambda^3) X^0 \wedge \alpha_3, & \Omega_{\lambda-1}^2 &= \lambda^3 X^2 \wedge \alpha_3 + (2\lambda - \lambda^2) X^0 \wedge \alpha_1, \\ \Omega_{\lambda-2}^3 &= \lambda^3 X^3 \wedge \alpha_1 - (2\lambda - \lambda^3) X^1 \wedge \alpha_3, & \Omega_{\lambda-1}^3 &= \lambda^3 X^3 \wedge \alpha_3 + (2\lambda - \lambda^3) X^1 \wedge \alpha_1. \end{aligned}$$

In the computation of $\Omega_{\lambda\nu}^\mu$ ($\mu, \nu \geq 0$), we use (2-4) and compare $\Omega_{\lambda\nu}^\mu$ and Ω_ν^μ . We have the following results, which should be understood modulo hyper-Kähler contribution, i.e., $\Omega_{\lambda\nu}^\mu$ should be understood modulo $\Omega_{\nu}^{\prime\mu}$:

$$\begin{aligned} \Omega_{\lambda 0}^0 &= \Omega_{\lambda 2}^2 = \Omega_0^0 - \lambda^2 (X^1 \wedge {}^tX^1 - X^3 \wedge {}^tX^3) \\ &= X^0 \wedge {}^tX^0 + X^1 \wedge {}^tX^1 + X^2 \wedge {}^tX^2 + X^3 \wedge {}^tX^3 - \lambda^2 (X^1 \wedge {}^tX^1 + X^3 \wedge {}^tX^3), \\ \Omega_{\lambda 1}^1 &= \Omega_{\lambda 3}^3 = \Omega_0^0 - \lambda^2 (X^0 \wedge {}^tX^0 - X^2 \wedge {}^tX^2) \\ &= X^0 \wedge {}^tX^0 + X^1 \wedge {}^tX^1 + X^2 \wedge {}^tX^2 + X^3 \wedge {}^tX^3 - \lambda^2 (X^0 \wedge {}^tX^0 + X^2 \wedge {}^tX^2) \end{aligned}$$

and

$$\begin{aligned}
\Omega_{\lambda_0^1} &= d(\Gamma_1 + \omega_1) - \lambda^2 d\alpha_1 + \lambda^2 (X^0 \wedge {}^t X^1 + X^2 \wedge {}^t X^3) \\
&\quad + (\Gamma_1 + \alpha_1 - \lambda^2 \alpha_1) \wedge \Gamma_0 + \Gamma_0 \wedge (\Gamma_1 + \alpha_1 - \lambda^2 \alpha_1) \\
&\quad + (-\Gamma_3 + \alpha_3 - \lambda^2 \alpha_3) \wedge (\Gamma_2 + \alpha_2) + (\Gamma_2 - \alpha_2) \wedge (\Gamma_3 + \alpha_3 - \lambda^2 \alpha_3) \\
&= \Omega_0^1 - \lambda^2 (d\alpha_1 - 2\alpha_2 \wedge \alpha_3) + \lambda^2 (X^0 \wedge {}^t X^1 + X^2 \wedge {}^t X^3) \\
&= X^1 \wedge {}^t X^0 - X^0 \wedge {}^t X^1 + X^3 \wedge {}^t X^2 - X^2 \wedge {}^t X^3 + 2(X^1 \wedge {}^t X^0 + {}^t X^2 \wedge X^3) \\
&\quad + \lambda^2 (X^0 \wedge {}^t X^1 + X^2 \wedge {}^t X^3 - 2{}^t X^1 \wedge X^0 - 2{}^t X^2 \wedge X^3) \\
\Omega_{\lambda_0^2} &= d(\Gamma_2 + \alpha_2) + \lambda^2 (X^3 \wedge {}^t X^1 - X^1 \wedge {}^t X^3) \\
&\quad + (\Gamma_2 + \alpha_2) \wedge \Gamma_0 + (\Gamma_3 - \alpha_3 + \lambda^2 \alpha_3) \wedge (\Gamma_1 + \alpha_1 - \lambda^2 \alpha_1) \\
&\quad + \Gamma_0 \wedge (\Gamma_2 + \alpha_2) + (-\Gamma_1 + \alpha_1 - \lambda^2 \alpha_1) \wedge (\Gamma_3 + \alpha_3 - \lambda^2 \alpha_3) \\
&= \Omega_0^2 + \lambda^2 (X^3 \wedge {}^t X^1 - X^1 \wedge {}^t X^3) + 2\lambda^2 \alpha_3 \wedge \alpha_1 \\
&= X^2 \wedge {}^t X^0 - X^0 \wedge {}^t X^2 + X^2 \wedge {}^t X^3 - X^3 \wedge {}^t X^1 + 2({}^t X^2 \wedge X^0 + {}^t X^3 \wedge X^1) \\
&\quad + \lambda^2 (X^3 \wedge {}^t X^1 - X^1 \wedge {}^t X^3) + 4\lambda^2 \alpha_3 \wedge \alpha_1 .
\end{aligned}$$

Similarly, modulo Ω''^μ_ν , we have :

$$\begin{aligned}
\Omega_{\lambda_0^3} &= \Omega_0^3 - \lambda^2 (d\alpha_3 - 2\alpha_1 \wedge \alpha_2) + \lambda^2 (-X^2 \wedge {}^t X^1 + X^0 \wedge {}^t X^3) \\
&= X^3 \wedge {}^t X^0 - X^0 \wedge {}^t X^3 + X^2 \wedge {}^t X^1 - X^1 \wedge {}^t X^2 + 2({}^t X^3 \wedge X^0 + {}^t X^1 \wedge X^2) \\
&\quad + \lambda^2 (-X^2 \wedge {}^t X^1 + X^0 \wedge {}^t X^3 - 2{}^t X^3 \wedge X^0 - 2{}^t X^1 \wedge X^2) , \\
\Omega_{\lambda_1^2} &= \Omega_1^2 + \lambda^2 (d\alpha_3 - 2\alpha_1 \wedge \alpha_2) + \lambda^2 (-X^3 \wedge {}^t X^0 + X^1 \wedge {}^t X^2) \\
&= X^3 \wedge {}^t X^0 - X^0 \wedge {}^t X^3 + X^2 \wedge {}^t X^1 - X^1 \wedge {}^t X^2 + 2({}^t X^0 \wedge X^3 + {}^t X^2 \wedge X^1) \\
&\quad + \lambda^2 (-X^3 \wedge {}^t X^0 + X^1 \wedge {}^t X^2 + 2{}^t X^3 \wedge X^0 + 2{}^t X^1 \wedge X^2) , \\
\Omega_{\lambda_1^3} &= \Omega_1^3 + \lambda^2 (X^2 \wedge {}^t X^0 - X^0 \wedge {}^t X^2) + 4\lambda^2 \alpha_3 \wedge \alpha_1 \\
&= -X^2 \wedge {}^t X^0 + X^0 \wedge {}^t X^2 - X^1 \wedge {}^t X^3 + X^3 \wedge {}^t X^1 + 2({}^t X^2 \wedge X^0 + {}^t X^3 \wedge X^1) \\
&\quad + \lambda^2 (X^2 \wedge {}^t X^0 - X^0 \wedge {}^t X^2) + 4\lambda^2 \alpha_3 \wedge \alpha_1 , \\
\Omega_{\lambda_2^3} &= \Omega_2^3 + \lambda^2 (d\alpha_1 - 2\alpha_2 \wedge \alpha_3) + \lambda^2 (X^2 \wedge {}^t X^3 + X^0 \wedge {}^t X^1) \\
&= X^1 \wedge {}^t X^0 - X^0 \wedge {}^t X^1 + X^3 \wedge {}^t X^2 - X^2 \wedge {}^t X^3 + 2(X^0 \wedge {}^t X^1 + {}^t X^3 \wedge X^2) \\
&\quad + \lambda^2 (X^2 \wedge {}^t X^3 + X^0 \wedge {}^t X^1 + 2{}^t X^1 \wedge X^0 + 2{}^t X^2 \wedge X^3) .
\end{aligned}$$

Here, we recall that

$$\begin{aligned}
\Omega_0^\mu &= \tilde{\Omega}_0^\mu + \Omega_0'^\mu \\
&= X^\mu \wedge {}^t X^0 - X^0 \wedge {}^t X^\mu + X^\nu \wedge {}^t X^\eta - X^\eta \wedge {}^t X^\nu \\
&\quad + 2({}^t X^\mu \wedge X^0 + {}^t X^\eta \wedge X^\nu) + \Omega_0''^\mu , \\
\Omega_\nu^\eta &= \tilde{\Omega}_\nu^\eta + \Omega_\nu'^\eta \\
&= -X^\mu \wedge {}^t X^0 + X^0 \wedge {}^t X^\mu - X^\nu \wedge {}^t X^\eta + X^\eta \wedge {}^t X^\nu \\
&\quad + 2({}^t X^\mu \wedge X^0 + {}^t X^\eta \wedge X^\nu) + \Omega_\nu''^\eta ,
\end{aligned}$$

(μ, η, ν) being any cyclic permutation of $(1, 2, 3)$, where the “hyper-Kähler part” Ω''^μ_ν has no contribution to the Ricci tensor. Therefore we can ignore the Ω''^μ_ν -part in the computation of the Ricci tensor.

We are now ready to compute the Ricci tensor of the metric g_λ^{can} on \mathcal{Z} . Note that the dependency on the point under consideration is completely hidden in the “hyper-Kähler part” Ω' and Ω' has no contribution to the Ricci tensor. Therefore, although we do not assume the homogeneity of M , we are able to compute the Ricci tensor purely Lie theoretically as if we were working on the Riemannian homogeneous space. Let $\xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \xi_2, \xi_3$ be the frame of \mathcal{Z} dual to the coframe $\alpha_1, \alpha_2, X^0, X^1, X^2, X^3$. Then

$$\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1}, \xi_0, \xi_1, \xi_2, \xi_3$$

is the frame (orthonormal w.r.to the metric g_λ^{can}) dual to the coframe

$$\lambda\alpha_1, \lambda\alpha_2, X^0, X^1, X^2, X^3.$$

We compute the components of the Ricci tensor using the following formula. Let $\{e_i\}_{i=1}^n$ and $(\Omega_j^i)_{i,j=1,\dots,n}$ be an orthonormal frame and the associated curvature form of an n -dimensional Riemannian manifold. Then we have

$$\text{Ric}(e_i, e_j) = \sum_{k=1}^n g(R(e_i, e_k)e_k, e_j) = \sum_{k=1}^n g(\Omega_k^j(e_i, e_k)e_j, e_j).$$

Using this formula, the components of the Ricci tensor $\text{Ric}_\lambda^{\text{can}}$ are computed as follows. If the expression $\Omega_{\lambda_k}^{-2}(\lambda^{-1}\xi_{-2}, \xi_k)\lambda^{-1}\xi_{-2}$ in the following computation means to take the summation over all combination of a fixed vector in ξ_{-2} and any vector in ξ_k ($0 \leq k \leq 3$), we have

$$\begin{aligned} \text{Ric}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-2}) &= g_\lambda^{\text{can}}(\text{Rm}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1})\lambda^{-1}\xi_{-1}, \lambda^{-1}\xi_{-2}) \\ &\quad + \sum_{k=0}^3 g_\lambda^{\text{can}}(\text{Rm}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-2}, \xi_k)\xi_k, \lambda^{-1}\xi_{-2}) \\ &= g_\lambda^{\text{can}}(\Omega_{\lambda^{-1}}^{-2}(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1})\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-2}) \\ &\quad + \sum_{k=0}^3 g_\lambda^{\text{can}}(\Omega_{\lambda_k}^{-2}(\lambda^{-1}\xi_{-2}, \xi_k)\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-2}) \\ &= \frac{4}{\lambda^2} + 4n\lambda^2. \end{aligned}$$

Indeed, $\Omega_{\lambda^{-1}}^{-2}(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1}) = 4\alpha_1 \wedge \alpha_3(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1}) = \frac{4}{\lambda^2}$ and

$\sum_{k=0}^3 \Omega_{\lambda_k}^{-2}(\lambda^{-1}\xi_{-2}, \xi_k) = \lambda^3 X^k \wedge \alpha_1(\xi_k, \lambda^{-1}\xi_{-2}) = 4n\lambda^2$. Similarly, we have

$$\text{Ric}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-1}, \lambda^{-1}\xi_{-1}) = \frac{4}{\lambda^2} + 4n\lambda^2.$$

Next, under the similar convention in the expression such as $\Omega_{\lambda_k}^{-1}(\lambda^{-1}\xi_{-2}, \xi_k)\lambda^{-1}\xi_{-1}$, we have

$$\begin{aligned} \text{Ric}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1}) &= g_\lambda^{\text{can}}(\text{Rm}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1})\lambda^{-1}\xi_{-1}, \lambda^{-1}\xi_{-1}) \\ &\quad + \sum_{k=0}^3 g_\lambda^{\text{can}}(\text{Rm}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-2}, \xi_k)\xi_k, \lambda^{-1}\xi_{-1}) \\ &= g_\lambda^{\text{can}}(\Omega_{\lambda_{-1}}^{-1}(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1})\lambda^{-1}\xi_{-1}, \lambda^{-1}\xi_{-1}) \\ &\quad + \sum_{k=0}^3 g_\lambda^{\text{can}}(\Omega_{\lambda_k}^{-1}(\lambda^{-1}\xi_{-2}, \xi_k)\lambda^{-1}\xi_{-1}, \lambda^{-1}\xi_{-1}) \\ &= 0. \end{aligned}$$

Indeed, $\sum_{k=0}^3 \Omega_{\lambda_k}^{-1}(\lambda^{-1}\xi_{-2}, \xi_k) = \lambda^3 X^0 \wedge \alpha_3 - (2\lambda - \lambda^2)X^2 \wedge \alpha_1(\xi_0, \lambda^{-1}\xi_{-2}) + \lambda^3 X^1 \wedge \alpha_3 - (2\lambda - \lambda^2)X^3 \wedge \alpha_1(\xi_1, \lambda^{-1}\xi_{-2}) + \lambda^3 X^2 \wedge \alpha_3 + (2\lambda - \lambda^2)X^0 \wedge \alpha_1(\xi_2, \lambda^{-1}\xi_{-2}) + \lambda^3 X^1 \wedge \alpha_3 - (2\lambda - \lambda^2)X^1 \wedge \alpha_1(\xi_3, \lambda^{-1}\xi_{-2}) = 0$.

Next, under the similar convention in the expression such as $\text{Rm}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-2}, \xi_k)\xi_k$, we have

$$\begin{aligned} \text{Ric}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-2}, \xi_0) &= g_\lambda^{\text{can}}(\text{Rm}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1})\lambda^{-1}\xi_{-1}, \xi_0) \\ &\quad + \sum_{k=0}^3 g_\lambda^{\text{can}}(\text{Rm}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-2}, \xi_k)\xi_k, \xi_0) = 0. \end{aligned}$$

Indeed, $\Omega_{\lambda_{-1}}^0(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1})\xi_0, \xi_0 = \lambda^3(X^0 \wedge \alpha_3 - (2\lambda - \lambda^2)X^2 \wedge \alpha_1)(\lambda^{-1}\xi_{-1}, \lambda^{-1}\xi_{-2})\xi_0 = 0$ and $\Omega_{\lambda_k}^0(\lambda^{-1}\xi_{-2}, \xi_k)\xi_0 = 0$. Similarly, we have

$$\begin{aligned} \text{Ric}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-2}, \xi_1) &= \text{Ric}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-2}, \xi_2) = \text{Ric}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-2}, \xi_3) \\ &= \text{Ric}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-1}, \xi_0) = \text{Ric}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-1}, \xi_1) = \text{Ric}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-1}, \xi_2) \\ &= \text{Ric}_\lambda^{\text{can}}(\lambda^{-1}\xi_{-1}, \xi_3) = 0. \end{aligned}$$

We use (2-6) and (2-10) to compute the remaining components of the Ricci tensor (under the similar convention in the expression such as $\text{Rm}_\lambda^{\text{can}}(\xi_0, \xi'_0)\xi'_0$ and $\text{Rm}_\lambda^{\text{can}}(\xi_0, \xi_k)\xi_k$ which is explained more precisely below) :

$$\begin{aligned} \text{Ric}_\lambda^{\text{can}}(\xi_0, \xi_0) &= g_\lambda^{\text{can}}(\text{Rm}_\lambda^{\text{can}}(\xi_0, \lambda^{-1}\xi_{-2})\lambda^{-1}\xi_{-2}, \xi_0) + g_\lambda^{\text{can}}(\text{Rm}_\lambda^{\text{can}}(\xi_0, \lambda^{-1}\xi_{-1})\lambda^{-1}\xi_{-1}, \xi_0) \\ &\quad + g_\lambda^{\text{can}}(\text{Rm}_\lambda^{\text{can}}(\xi_0, \xi'_0)\xi'_0, \xi_0) + \sum_{k=1}^3 g_\lambda^{\text{can}}(\text{Rm}_\lambda^{\text{can}}(\xi_0, \xi_k)\xi_k, \xi_0) \\ &= 4n + 8 - 4\lambda^2. \end{aligned}$$

Indeed, $[\text{Rm}_\lambda^{\text{can}}(\xi_0, \lambda^{-1}\xi_{-2})\lambda^{-1}\xi_{-2}]_{\xi_0} = \Omega_{\lambda_{-2}}^0(\xi_0, \lambda^{-1}\xi_{-2})\xi_0 = \lambda^3 X^0 \wedge \alpha_1(\xi_0, \lambda^{-1}\xi_{-2})\xi_0 = \lambda^2 \xi_0$, $[\text{Rm}_\lambda^{\text{can}}(\xi_0, \lambda^{-1}\xi_{-1})\lambda^{-1}\xi_{-1}]_{\xi_0} = \Omega_{\lambda_{-1}}^0(\xi_0, \lambda^{-1}\xi_{-1})\xi_0 = \lambda^3 X^0 \wedge \alpha_3(\xi_0, \lambda^{-1}\xi_{-1})\xi_0 = \lambda^2 \xi_0$. In the following computation, as in the above computation, we fix a vector in ξ_0 and sum over all vectors in ξ'_0 (the same as ξ_0 but the fixed vector in ξ_0 removed) or those in ξ_k . In other words, the pair (ξ_0, ξ'_0) which appears in the following expression, e.g., $\text{Rm}_\lambda^{\text{can}}(\xi_0, \xi'_0)\xi'_0$ (resp $\text{Rm}_\lambda^{\text{can}}(\xi_0, \xi_k)\xi_k$)

means to take the sum over all combination of the fixed vector in ξ_0 and vectors in ξ_0 except the fixed one (resp. vectors in ξ_k). We then have

$$[\text{Rm}_\lambda^{\text{can}}(\xi_0, \xi'_0)\xi'_0]_{\xi_0} = \Omega_{\lambda_0'}^0(\xi_0, \xi'_0)\xi_0 = (n-1)\xi_0$$

and furthermore we have

$$\begin{aligned} & [\sum_{k=1}^3 \text{Rm}_\lambda^{\text{can}}(\xi_0, \xi_k)\xi_k]_{\xi_0} = \sum_{k=1}^3 \Omega_{\lambda_k}^0(\xi_0, \xi_k)\xi_0 \\ &= \{X^1 \wedge {}^tX^0 - X^0 \wedge {}^tX^1 + X^3 \wedge {}^tX^2 - X^2 \wedge {}^tX^3 + 2(X^1 \wedge {}^tX^0 + {}^tX^2 \wedge X^3) \\ &\quad + \lambda^2(X^0 \wedge {}^tX^1 + X^2 \wedge {}^tX^3 - 2{}^tX^1 \wedge X^0 - 2{}^tX^2 \wedge X^3)\}(\xi_1, \xi_0)\xi_0 \\ &\quad + \{X^2 \wedge {}^tX^0 - X^0 \wedge {}^tX^2 + X^2 \wedge {}^tX^3 - X^3 \wedge {}^tX^1 + 2({}^tX^2 \wedge X^0 + {}^tX^3 \wedge X^1) \\ &\quad + \lambda^2(X^3 \wedge {}^tX^1 - X^1 \wedge {}^tX^3) + 4\lambda^2\alpha_3 \wedge \alpha_1\}(\xi_2, \xi_0)\xi_0 \\ &\quad + \{X^3 \wedge {}^tX^0 - X^0 \wedge {}^tX^3 + X^2 \wedge {}^tX^1 - X^1 \wedge {}^tX^2 + 2({}^tX^3 \wedge X^0 + {}^tX^1 \wedge X^2) \\ &\quad + \lambda^2(-X^2 \wedge {}^tX^1 + X^0 \wedge {}^tX^3 - 2{}^tX^3 \wedge X^0 - 2{}^tX^1 \wedge X^2)\}(\xi_3, \xi_0)\xi_0 \\ &= \{(n+3-3\lambda^2) + (n+3) + (n+3-3\lambda^2)\}\xi_0 \\ &= (3n+9-6\lambda^2)\xi_0 \end{aligned}$$

where $[\cdots]_{\xi_i}$ in the above expression means to take the component of the fixed vector in ξ_0 of \cdots . Therefore we have

$$\text{Ric}_\lambda^{\text{can}}(\xi_0, \xi_0) = 2\lambda^2 + (n-1) + 3n + 9 - 6\lambda^2 = 4n + 8 - 4\lambda^2.$$

Furthermore, we have

$$\begin{aligned} \text{Ric}_\lambda^{\text{can}}(\xi_0, \xi_1) &= g_\lambda^{\text{can}}(\text{Rm}_\lambda^{\text{can}}(\xi_0, \lambda^{-1}\xi_{-2})\lambda^{-1}\xi_{-2}, \xi_1) + g_\lambda^{\text{can}}(\text{Rm}_\lambda^{\text{can}}(\xi_0, \lambda^{-1}\xi_{-1})\lambda^{-1}\xi_{-1}, \xi_1) \\ &\quad + \sum_{k=1}^3 g_\lambda^{\text{can}}(\text{Rm}_\lambda^{\text{can}}(\xi_0, \xi_k)\xi_k, \xi_1) = 0. \end{aligned}$$

Indeed, $[\text{Rm}_\lambda^{\text{can}}(\xi_0, \lambda^{-1}\xi_{-2})\xi_{-2}]_{\xi_1} = \Omega_{\lambda_{-2}}^1(\xi_0, \lambda^{-1}\xi_{-2})\xi_1 = (\lambda^3X^1 \wedge \alpha_1 + (2\lambda - \lambda^2)X^3 \wedge \alpha_3)(\xi_0, \lambda^{-1}\xi_{-2})\xi_1 = 0$, $[\text{Rm}_\lambda^{\text{can}}(\xi_0, \lambda^{-1}\xi_{-2})\xi_{-2}]_{\xi_1} = \Omega_{\lambda_{-1}}^1(\xi_0, \lambda^{-1}\xi_{-1})\xi_1 = (\lambda^3X^1 \wedge \alpha_3 - (2\lambda - \lambda^2)X^3 \wedge \alpha_1)(\xi_0, \lambda^{-1}\xi_{-1})\xi_1 = 0$ and $\text{Rm}_\lambda^{\text{can}}(\xi_0, \xi_k)\xi_k = \Omega_{\lambda_k}^1(\xi_0, \xi_k)\xi_1 = 0$.

Similarly we have

$$\begin{aligned} \text{Ric}_\lambda^{\text{can}}(\xi_1, \xi_1) &= \text{Ric}_\lambda^{\text{can}}(\xi_2, \xi_2) = \text{Ric}_\lambda^{\text{can}}(\xi_3, \xi_3) = 4n + 8 - 4\lambda^2, \\ \text{Ric}_\lambda^{\text{can}}(\xi_0, \xi_2) &= \text{Ric}_\lambda^{\text{can}}(\xi_0, \xi_3) = \text{Ric}_\lambda^{\text{can}}(\xi_1, \xi_2) = \text{Ric}_\lambda^{\text{can}}(\xi_1, \xi_3) = \text{Ric}_\lambda^{\text{can}}(\xi_2, \xi_3) = 0. \end{aligned}$$

To sum up, we have

Proposition 2.4. *The Ricci tensor of the metric*

$$g_\lambda^{\text{can}} = \lambda^2(\alpha_1^2 + \alpha_3^2) + \sum_{i=0}^3 {}^tX^i \cdot X^i$$

on the twistor space \mathcal{Z} is given by the formula

$$(2-15) \quad \text{Ric}_\lambda^{\text{can}} = (4\lambda^{-2} + 4n\lambda^2)\lambda^2(\alpha_1^2 + \alpha_3^2) + (4n + 8 - 4\lambda^2)({}^tX^0 \cdot X^0 + {}^tX^1 \cdot X^1 + {}^tX^2 \cdot X^2 + {}^tX^3 \cdot X^3) .$$

In particular, the canonical deformation metric g_λ^{can} is Einstein if and only if $\lambda^2 = 1$ or $\lambda^2 = \frac{1}{n+1}$. If $\lambda^2 = 1$ then g_λ^{can} is Kähler-Einstein and if $\lambda^2 = \frac{1}{n+1}$ then g_λ^{can} is Einstein, Hermitian but not Kähler.

§3. Z-metrics.

In the following discussion, we introduce new metrics (called Z-metrics denoted by g_λ^Z) on the twistor space of positive quaternion Kähler manifolds and compute the curvature form. We use the moving frame computation. The scalings of the various standard metrics are hidden in the computation. To avoid confusion, we fix our scaling convention in the following way :

- We fix the scale of the invariant metric of $\mathbb{H}\mathbb{P}^n$ so that the sectional curvatures range in the interval $[1, 4]$, i.e., $\text{Ric}(g_{\mathbb{H}\mathbb{P}^n}) = 4(n+2)g_{\mathbb{H}\mathbb{P}^n}$ and so $\text{Scal}(g_{\mathbb{H}\mathbb{P}^n}) = 16n(n+2)$. Namely we set

$$\tilde{S} = 16n(n+2)$$

from here on.

- We fix the scale of the Fubini-Study metric of the \mathbb{P}^1 -fiber of the twistor fibration and other cases so that the Gaussian curvature is identically 4.
- We fix the scale of the invariant metric of the $\text{Sp}(1)$ -fiber of the extended twistor fibration $\tilde{\mathcal{Z}} \rightarrow M$ so that the sectional curvature is identically 1.

We will consider the following partial scaling similar to the case of canonical deformation metrics in §2.

- We normalize the base quaternion Kähler metric g so that $S = \tilde{S}$ holds and scale the $(\alpha_1 - \alpha_1(\xi_1)X^1 - \alpha_1(\xi_3)X^3)^2 + (\alpha_3 - \alpha_3(\xi_1)X^1 - \alpha_3(\xi_3)X^3)^2$ -part (ignoring “invisible” part) by the parameter λ^2 (so that the “curvature becomes λ^{-2} -times the original one in this direction”). It turns out that the Z-metric g_λ^Z is Einstein (but not Kähler) on \mathcal{Z} if and only if $\lambda = \frac{1}{n+2}$.

Next we proceed to a construction of a new family of Riemannian metrics

$$\mathbb{R}_+ \cdot \mathcal{F}^Z = \{\rho g_\lambda^Z\}_{\lambda>0, \rho>0}$$

(the family of Z-metrics) on the twistor space \mathcal{Z} . Our strategy is to modify the construction of the canonical deformation metrics so that kill the $-\lambda^2$ -term in before the $\sum_{i=0}^3 {}^tX^i \cdot X^i$ -term in the formula of $\text{Ric}_\lambda^{\text{can}}$ in Proposition 2.4. Indeed, the $-\lambda^2$ -term in question constitutes the basic reason why the family of canonical deformation metrics is *not* a Ricci flow unstable cell. The simplest example of a Ricci flow unstable cell arises in the Ricci flow solutions on the product space $S^n \times S^n$ where initial metric an independently scaled constant curvature metrics (if the initial metric has slightly different curvature, then the S^n -factor with larger curvature will extinct earlier and before its extinction the metric of the total product space is “far” from being Einstein !). Therefore we try to modify the construction of the canonical deformation metric so that the resulting metric becomes “closer” to the “product” metric (of (M, g) and $(\mathbb{P}^1, g_{\text{FS}})$, in a “weak”

sense). The construction should be “canonically” based on the definition of the twistor space in a way somewhat different from the family of canonical deformation metrics. So we begin the construction of \mathcal{F}^Z with a useful interpretation of the twistor space. We identify \mathbb{H} with \mathbb{C}^2 by writing $x_0 + ix_1 + jx_2 + kx_3 = x_0 + jx_2 + i(x_1 + jx_3)$. Let $x + jy \in \mathbb{H}$ and $u + jv \in \mathbb{H}$. The manipulation

$$(x + jy)(u + jv) = (xu - \bar{y}v) + j(yu + \bar{x}v) = \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

implies that the right action of the unit quaternions is identified with the standard holomorphic action of $SU(2)$ to \mathbb{C}^2 and the left action of the unit quaternions is not holomorphic. The right action of the unit quaternion $u + jv$ is holomorphic if and only if $v = 0$, i.e., $u + jv$ is the unit complex number. The local decomposition $SO(4) = Sp(1)_l \times Sp(1)_r$ implies that the standard action of $SO(4)$ on \mathbb{R}^4 decomposes into the left and right actions of $Sp(1)$ (denoted by $Sp(1)_l$ and $Sp(1)_r$). Therefore the space of the orthogonal complex structures of \mathbb{H} is identified with $SO(4)/U(2) \cong Sp(1)_r/U(1) = \mathbb{P}^1$. On the other hand, the standard action of $Sp(1)_l = SU(2)$ on \mathbb{C}^2 implies that the space of all complex lines in \mathbb{C}^2 is identified with $Sp(1)_l/U(1) = SU(2)/S(U(1) \times U(1)) = \mathbb{P}^1$. The identification of \mathbb{P}^1 with the space of all orthogonal complex structures on \mathbb{H} is given by

$$\begin{aligned} & \text{orthogonal complex structure } J \text{ on } \mathbb{H} \\ & \leftrightarrow U(1)\text{-subgroup of } Sp(1)_r \text{ which acts on } \mathbb{C}_J^2 \text{ holomorphically} \end{aligned}$$

and the identification of \mathbb{P}^1 with the space of all complex lines in \mathbb{C}^2 is given by

$$\begin{aligned} & \text{complex line } L \text{ in } \mathbb{C}^2 \\ & \leftrightarrow U(1)\text{-subgroup of } Sp(1)_l \text{ which fixes } [L] \in \mathbb{P}^1 . \end{aligned}$$

Identifying $Sp(1)_l$ and $Sp(1)_r$, we can transfer a $U(1)$ -subgroup in $Sp(1)_l$ to a $U(1)$ -subgroup in $Sp(1)_r$. A $U(1)$ -subgroup in $Sp(1)_r$ determines an axis of rotation of $\mathbb{P}^1(\mathbb{C})$ and therefore two lines in \mathbb{C}^2 fixed by the $U(1)$ -action. As the twistor line $\mathbb{P}^1(\mathbb{C})$ is simply connected, we can choose one of them globally and therefore we have the following canonical correspondence

$$\begin{aligned} & \text{orthogonal complex structure } J \text{ on } \mathbb{H} \\ & \leftrightarrow \text{a complex line } L_J \text{ of } \mathbb{C}_J^2 . \end{aligned}$$

Similarly, $Sp(n)Sp(1)/Sp(n)Sp(1) \cap U(2n) = \mathbb{P}^1$ is identified with the space of all orthogonal complex structures of \mathbb{H}^n and also to the space of all (simultaneous choice of) complex lines in each \mathbb{H} -line in \mathbb{H}^n . Therefore, applying the above correspondence to each \mathbb{H} -lines in \mathbb{H}^n , we have the following canonical correspondence

$$\begin{aligned} & \text{orthogonal complex structure } J \text{ on } \mathbb{H}^n \\ & \leftrightarrow \text{a real } (4n - 2)\text{-dimensional complex subspace } D'_J \text{ of } \mathbb{C}_J^{2n} \\ & \text{which cuts out a complex line from each } \mathbb{H}\text{-line in } \mathbb{H}^n \\ & \leftrightarrow \text{a complex line } L_J = (D'_J)^\perp \text{ of } \mathbb{C}_J^{2n} . \end{aligned}$$

Consider the standard twistor fibration

$$\pi : (\mathbb{P}^{2n+1}(\mathbb{C}), \mathbb{P}^{2n-1}(\mathbb{C})) \rightarrow (\mathbb{P}^n(\mathbb{H}), \mathbb{P}^{n-1}(\mathbb{H})) .$$

Pick a point $m \in \mathbb{P}^n(\mathbb{H}) - \mathbb{P}^{n-1}(\mathbb{H})$. Then the fiber $\mathbb{P}_m^1 = \pi^{-1}(m)$ is identified with the moduli space of the orthogonal complex structures of $T_m \mathbb{P}^n(\mathbb{H}) \cong \mathbb{H}^n$. Pick a point $z \in \mathbb{P}_m^1$ and let $J(z)$ be the corresponding orthogonal complex structure. The hyperplane D_z of $\mathbb{P}^{2n+1}(\mathbb{C})$ generated by $\mathbb{P}^{2n-1}(\mathbb{C}) = \pi^{-1}(\mathbb{P}^{n-1}(\mathbb{H}))$ (inverse image of the hyperplane at infinity) and $z \in \mathbb{P}_m^1$ gives an identification of the affine part $\mathbb{H}^n \cong \mathbb{P}^n(\mathbb{H}) - \mathbb{P}^{n-1}(\mathbb{H})$ with $\mathbb{C}_{J(z)}^{2n}$, i.e., \mathbb{H}^n with the orthogonal complex structure $J(z)$. Assume that the point m under consideration is the origin of \mathbb{H}^n . Then the tangent space of D'_z coincides with the horizontal subspace H_z at $z \in \mathbb{P}_m^1$. At each point $z \in \mathbb{P}_m^1$ we have a 2-dimensional subspace $L_{J(z)} := (D'_{J(z)})^\perp$ of $D_z = H_z$ which is a complex line w.r.to the orthogonal complex structure $J(z)$. If we consider \mathbb{H} -lines $L \cong \mathbb{P}^1(\mathbb{H})$ in $\mathbb{P}^n(\mathbb{H})$ passing through the origin $m \in \mathbb{H}^n = \mathbb{P}^n(\mathbb{H}) - \mathbb{P}^{n-1}(\mathbb{H})$ and the sub twistor fibrations $\mathbb{P}^3(\mathbb{C}) \cong \pi^{-1}(L) \rightarrow L \cong \mathbb{P}^1(\mathbb{H})$ over these \mathbb{H} -lines, the collection of $J(z)$ -complex lines from each \mathbb{H} -line in \mathbb{H}^n constitutes a real $(4n-2)$ -dimensional subspace D'_z which is a complex subspace of $\mathbb{C}_{J(z)}^{2n}$. From this consideration, we see that there exists a column n vector $\xi_0 \in D'_z$ uniquely modulo $\text{SO}(2)$ -rotation so that $\{\xi_0, J(z)\xi_0\}_{\text{span}}$ is a real $(2n)$ -dimensional complex subspace in D'_z . We observe that there is a “canonical” way constructing a real $(4n)$ -dimensional non-horizontal subspace D_z of $T_z \mathbb{P}^{2n+1}(\mathbb{C})$ which is complex w.r.to $J(z)$ by using the orthogonal complement $L_{J(z)} = (D'_{J(z)})^\perp$. In fact, D_z is constructed as a direct sum of D'_z and a complex line $L'_{J(z)}$ in the linear subspace ($\cong \mathbb{C}^2$) of $T_z \mathbb{P}^{2n+1}(\mathbb{C})$ spanned by $T_z \mathbb{P}_m^1$ and $L_{J(z)}$. We would like to “canonically” construct $L'_{J(z)}$ by choosing unit vectors $u \in T_z \mathbb{P}_m^1$ and $v \in L_{J(z)}$ and putting $L'_{J(z)} := \mathbb{C} \cdot (u + v)$. Here, u and v are chosen from the tangent bundle $\mathcal{O}(2)$ of \mathbb{P}^1 . Therefore defining such $L'_{J(z)}$ is equivalent to defining a non-trivial embedding $\mathcal{O}(2) \rightarrow \mathcal{O}(2) \oplus \mathcal{O}(2)$, i.e., defining a graph of a non-zero section of $\text{Hom}(\mathcal{O}(2), \mathcal{O}(2) \oplus \mathcal{O}(2)) = \mathcal{O}(-2) \otimes \mathcal{O}(2) = \mathcal{O}$. Therefore $L'_{J(z)}$ is defined modulo unit complex numbers, i.e., a choice of a unit non-zero section of \mathcal{O} . So, we choose $1 \in H^0(\mathbb{P}^1, \mathcal{O})$ to define $L'_{J(z)}$ canonically. The S^1 -fiber in the extended twistor space $\tilde{\mathcal{Z}} = \frac{\text{Sp}(n+1)/\mathbb{Z}_2}{\text{Sp}(n)\text{Sp}(1) \cap \text{SU}(2n)}$ sitting over a point $z \in \mathcal{Z}$ of a twistor line equip $L_{J(z)}$ (also $L'_{J(z)}$) with additional information, i.e., an oriented orthonormal frame. Indeed, the S^1 -bundle $\tilde{\mathcal{Z}} \rightarrow \frac{\text{Sp}(n+1)/\mathbb{Z}_2}{\text{Sp}(n)\text{Sp}(1) \cap \text{U}(2n)}$ induces the Hopf fibration $S^3 \rightarrow S^2$ over each twistor line in \mathcal{Z} and therefore the S^1 -fiber over z corresponds to the rotation by unit complex numbers of an oriented orthonormal frame of $L_{J(z)}$ (also $L'_{J(z)}$). Therefore we have a well-defined

$$L'_{J(z)} := \mathbb{C} \cdot (u + v)$$

which is “canonical” from Riemannian view point (indeed, this is also “canonical” from the construction of Z-metrics which is explained in the following). We get a “canonical” non-horizontal subspace D_z as an orthogonal direct sum

$$D_z := D'_z \oplus L'_{J(z)}.$$

We proceed to a general case. Let (M^{4n}, g) ($n \geq 2$) be a positive quaternion Kähler manifold. Let \mathcal{H} be the horizontal distribution of the twistor fibration, i.e., the $(4n)$ -dimensional distribution consisting of the horizontal subspaces \mathcal{H}_z at $z \in \mathcal{Z}$. We consider the $\text{G}(4n-2, 4n)$ -bundle ($\text{G}(p, q)$ being the Grassmannian of p -dimensional subspaces in \mathbb{R}^q) associated to the holonomy reduction $\mathcal{P} \rightarrow M$ of the oriented orthonormal frame bundle. As the structure group of the oriented orthonormal frame bundle reduces from $\text{SO}(4n)$ to $\text{Sp}(n)\text{Sp}(1)$, the associated $\text{G}(4n-2, 4n)$ -bundle also reduces to a smaller bundle with fiber $\text{Sp}(n)\text{Sp}(1)/\text{Sp}(n)\text{Sp}(1) \cap \text{S}(\text{O}(2) \times \text{O}(4n-2)) \cong \text{Sp}(n)\text{Sp}(1)/\text{Sp}(n)\text{Sp}(1) \cap \text{U}(2n) \cong \mathbb{P}^1$ which turns out to be isomorphic to the twistor fibration

$\mathcal{Z} \rightarrow M$. Let $m \in M$ and $z \in \mathbb{P}_m^1 = \pi^{-1}(m)$, $\pi : \mathcal{Z} \rightarrow M$ being the twistor fibration. The holonomy reduction of the associated $G(4n-2, 4n)$ -bundle corresponds to the association

$$\mathbb{P}_m^1 \ni z \longmapsto \mathcal{D}'_z \in G(4n-2, H_z)$$

where \mathcal{D}'_z is the same as in the model case $\mathbb{P}^{2n+1}(\mathbb{C}) \rightarrow \mathbb{P}^n(\mathbb{H})$, i.e., \mathcal{D}'_z is a real $(4n-2)$ -dimensional subspace of $H_z \cong \mathbb{C}_{J(z)}^{2n}$ which is complex w.r.to $J(z)$ and specifies the complex line from each \mathbb{H} -lines in H_z ($J(z)$ is the orthogonal complex structure of $T_m M$ determined by $z \in \mathbb{P}_m^1$). Let $\mathcal{L}_{J(z)}$ be the orthogonal complement in \mathcal{H}_z of $\mathcal{D}'_z \subset \mathcal{H}_z$. Then $\mathcal{L}_{J(z)}$ is a complex line in $\mathbb{C}_{J(z)}^{2n}$ (\mathcal{H}_z equipped with the orthogonal complex structure $J(z)$). We consider the subspace ($\cong \mathbb{C}^2$) of $T_z \mathcal{Z}$ spanned by $T_z \mathbb{P}_m^1 \cong \mathbb{C}$ and $\mathcal{L}_{J(z)} (\cong \mathbb{C}) \subset \mathcal{H}_z$. If we choose unit vectors $u \in T_z \mathbb{P}_m^1$ and $v \in \mathcal{L}_{J(z)}$ in a suitable way, then

$$\mathcal{L}'_{J(z)} := \mathbb{C} \cdot (u + v)$$

is well-defined as before. Indeed, we define the extended twistor space by

$$\tilde{\mathcal{Z}} := \mathcal{P} / \mathrm{Sp}(n) \mathrm{Sp}(1) \cap \mathrm{SU}(n) .$$

Then the ambiguity by multiplication by unit complex numbers in the choice of u and v is interpreted as the rotation along the S^1 -fiber of $\tilde{\mathcal{Z}} \rightarrow \mathcal{Z}$. Therefore we can work on $\tilde{\mathcal{Z}}$ without ambiguity and the resulting vector $u + v$ on \mathcal{Z} is defined modulo multiplication by unit complex numbers. This means that the complex line $\mathbb{C} \cdot (u + v)$ is well-defined. The $(4n)$ -dimensional non-horizontal distribution $\mathcal{D} := \{\mathcal{D}_z\}_{z \in \mathcal{Z}}$ on \mathcal{Z} is now defined by

$$\mathcal{D}_z := \mathcal{D}'_z \oplus \mathcal{L}'_{J(z)} .$$

We choose a column n vector ξ_0 of \mathcal{D}'_z (representing an n -dimensional subspace of $T_m M$). Then $\{\xi_0, J(z)\xi_0\}_{\mathrm{span}}$ is a $J(z)$ -invariant $(2n)$ -dimensional subspace of \mathcal{D}'_z . In this situation, $J(z)$ is the unique orthogonal complex structure of $T_m M$ with the above property (i.e., any $(2n)$ -dimensional subspace of $T_m M$ spanned by $\{X_1, qX_1\}$ where $X_1 \in \mathcal{D}'_z$ and q is any orthogonal complex structure of $T_m M$ not equal to $\pm J(z)$ is not contained in \mathcal{D}'_z). The correspondence $z \mapsto \mathcal{D}'_z$ defines a $(4n-2)$ -dimensional horizontal distribution \mathcal{D}' on the twistor space \mathcal{Z} . Each point $z \in \mathcal{Z}$ represents an orthogonal complex structure $J(z)$ ($J(z) = J$, say) and a $(4n-2)$ -dimensional subspace $\mathcal{D}'_z \subset T_m M$ (z lies over $m \in M$) which contains the $(2n)$ -dimensional J -complex subspace spanned by $\{\xi_0, J\xi_0\}$. Let $\mathcal{S} \subset \mathrm{End}^{\mathrm{skew}}(TM)$ be the 3-dimensional sub-bundle over M which defines the quaternion Kähler structure of (M, g) . Let $\{I, J, K\}$ ($J = J(z)$ under consideration) be an orthonormal basis of \mathcal{S}_m . Then $\{\xi_0, I\xi_0, J\xi_0, K\xi_0\}$ is an orthonormal basis of $T_m M$. We extend ξ_0 to a vector field germ at $m \in M$ so that the contribution from the Levi-Civita connection to $\nabla \xi_0$ satisfies certain condition to be specified later. If we further extend the triple $\{I, J, K\}$ as orthonormal frame germ of the bundle $\mathcal{S} \rightarrow M$, we get an orthonormal frame germ $\{\xi_0, I\xi_0, J\xi_0, K\xi_0\}$. First we extend J to a unit section germ of the bundle $\mathcal{S} \rightarrow M$ so that ∇J vanishes in ξ_0 - and $J\xi_0$ -directions at m (this is the condition on the $\mathrm{sp}(1)$ -part of the Levi-Civita connection). Then we use the distribution \mathcal{D} which we have just constructed above to extend I and K . Namely we extend I and K in the following way. The section germ $\{\xi_0, I\xi_0, J\xi_0, K\xi_0\}$ of the holonomy reduction $\mathcal{P} \rightarrow M$ of the oriented orthonormal frame bundle composed with the projection $\mathcal{P} \rightarrow \mathcal{Z}$ defines a section germ σ of the twistor fibration $\pi : \mathcal{Z} \rightarrow M$ at m passing through $z \in \mathbb{P}_m^1$. Since σ is a map

germ $(M, m) \rightarrow (\mathcal{Z}, z)$ which is linearly non degenerate at m , we can define the $(4n)$ -dimensional subspace $(d\sigma)_m(T_m M) \subset T_z \mathcal{Z}$. We choose the extension of I and K so that the relation

$$(3-1) \quad (d\sigma)_m(T_m M) = \mathcal{D}_z$$

holds. Since local orthonormal frame fields of M one to one correspond to local sections of the bundle $\mathcal{P} \rightarrow M$, the requirement (3-1) is realized by certain extensions of I and K such that the triple $\{I, J, K\}$ constitutes an orthonormal frame germ of $\mathcal{S} \rightarrow M$ at m . Now we describe the procedure precisely. Given a section germ J satisfying $J^2 = -1$ of the bundle $\mathcal{S} \rightarrow M$, the extension of I and K so that the triple $\{I, J, K\}$ constitutes an oriented orthonormal frame germ of \mathcal{S} (which necessarily satisfies the quaternion relations) is unique modulo (non constant) $\text{SO}(2)$ -rotation. This way we get an orthonormal frame germ $\{\xi_0, I\xi_0, J\xi_0, K\xi_0\}$. This triple defines a section germ of the holonomy reduction $\mathcal{P} \rightarrow M$ of the oriented orthonormal frame bundle at m . The composition with the projection $\mathcal{P} \rightarrow \mathcal{Z}$ defines a section germ σ of the twistor bundle $\mathcal{Z} \rightarrow M$. The section germ σ defines a section germ (denoted by the same symbol σ) from (the image of σ in) \mathcal{Z} to \mathcal{P} and defines the pull-back of differential forms on \mathcal{P} to those on the twistor space \mathcal{Z} (taking values only on tangent vectors of the image of σ in \mathcal{Z}). The requirement that the section germ σ satisfies the condition (3-1) is equivalent to requiring the following conditions: (i) the section germ ξ_0 at m satisfies the condition that the contribution to $\nabla \xi_0$ at m from the components $\Gamma_0, \Gamma_2, \alpha_1$ and α_3 of the Levi-Civita connection of g vanish (are “invisible”) in ξ_0 - and $J\xi_0$ -directions⁸, (ii) the unit section germ J of the bundle $\mathcal{S} \rightarrow M$ at m satisfies the condition that the contribution to ∇J at m vanishes in ξ_0 - and $J\xi_0$ -directions, (iii) the $\text{sp}(1)$ -part of the Levi-Civita connection in ∇I and ∇K at m vanishes in ξ_0 - and $J\xi_0$ -directions, (iv) the original $\text{sp}(1)$ -part of the connection form defined on \mathcal{Z} should be replaced by the pull-back of the $\text{sp}(1)$ -part of the connection form defined on \mathcal{P} via the map germ $\mathcal{Z} \rightarrow \mathcal{P}$ constructed from the section germ (composed with the projection $\mathcal{P} \rightarrow \mathcal{Z}$) $\sigma : (M, m) \rightarrow (\mathcal{Z}, z)$ satisfying (3-1). However, what we get by this procedure coincides with the original one.

As we will work on the twistor space \mathcal{Z} , we must define ξ_i ’s as tangent vectors at z of \mathcal{Z} and extend them as a vector field germs on \mathcal{Z} (rather than vector fields germs on M). We recall that ξ_i ’s, as tangent vectors at $z \in \mathcal{Z}$, are defined as the image under the differential $d\sigma$ of the orthonormal section germ σ determined by $\{\xi_0, I\xi_0, J\xi_0, K\xi_0\}$. Therefore if we extend ξ_0 as a section of the distribution $\{D'_z\}_{z \in \mathbb{P}_m^1}$ in the \mathbb{P}_m^1 -fiber direction, the extended object defines a desired vector field germ ξ_0 on \mathcal{Z} . It follows from the expression (2-2) that the derivation formula satisfied by the extended ξ_0 is expressed as

$$(3-2) \quad \nabla \xi_0 = \Gamma_0 \otimes \xi_0 + (\Gamma_1 + \alpha_1) \otimes \xi_1 + (\Gamma_2 + \alpha_2) \otimes \xi_2 + (\Gamma_3 + \alpha_3) \otimes \xi_3$$

where ∇ is the connection of the distribution \mathcal{D} on \mathcal{Z} obtained by pulling back the Levi-Civita connection of g on M via the local map σ from \mathcal{Z} to \mathcal{P} (in the \mathbb{P}^1 -fiber direction the formula (3-2) describes the rotation by the $\text{sp}(1)$ -part α_1 and α_3 of the Levi-Civita connection of g). Moreover, all 1-forms appearing as coefficients in the above formula are obtained by pulling back the connection form of the Levi-Civita connection of g . The connection form is originally defined on \mathcal{P} and pulled back via the section germ σ to \mathcal{Z} , where σ is regarded as a map from \mathcal{Z} to \mathcal{P} by identifying TM and $d\sigma(TM)$ and the extension of ξ_0 in the fiber \mathbb{P}_m^1 direction. Once we extend ξ_0 to a vector field

⁸ Although performing the projection $\mathcal{P} \rightarrow \mathcal{Z}$ significantly decreases the information on the Γ -part, the vanishing of Γ_0 and Γ_2 at m in the ξ_0 - and $J\xi_0$ -directions is still a very important information on the choice of ξ_0 , because this reflects the geometry behind the construction of the distribution \mathcal{D}' .

germ on \mathcal{Z} at z , we automatically get the quadruple $\{\xi_0, I\xi_0, J\xi_0, K\xi_0\}$ which defines a map germ from \mathcal{Z} to \mathcal{P} at z . We note the followings : (i) the components Γ_0 and Γ_2 from the Γ -part (the $\mathrm{Sp}(n)$ -component of the Levi-Civita connection of g) are “invisible” in ξ_0 - and $J\xi_0$ -directions (but “visible” in the ξ_1 - and ξ_3 - directions) at m and no condition is imposed on the Γ -part, (ii) the components α_1 and α_3 from the $\mathrm{sp}(1)$ -part of the Levi-Civita connection of g are “invisible” in ξ_0 - and $J\xi_0$ -directions (but “visible” in the ξ_1 - and ξ_3 - directions). In addition to these conditions, we note that (iii) the component α_2 is “invisible” at m in every direction (this is a consequence of the definitions of the twistor space and the triple $\{\alpha_1, \alpha_2, \alpha_3\}$).

The reason why no condition is imposed on the Γ -part at this stage is that the information on the Γ -part of the Levi-Civita connection form of g is lost after performing the projection $\mathcal{P} \rightarrow \mathcal{Z}$. This means that, although the equations $\alpha_1 = \alpha_3 = 0$ defines the horizontal subspace in the twistor space \mathcal{Z} , the Γ -part of the connection form which appears in $\nabla\xi_0$ depends on the original local section of the orthonormal frame bundle $\mathcal{P} \rightarrow M$ by which the Γ -part is pulled back to \mathcal{Z} from \mathcal{P} and therefore Γ_i ’s do not necessarily vanish at horizontal vectors in the twistor fibration $\mathcal{Z} \rightarrow M$. Later, we need to impose a certain condition on the Γ -part in question, in order to construct an Einstein metric together with Ricci flow unstable cell in §4 and effectively apply Bando-Shi’s gradient estimate in §5. The condition we need is the following :

$$(3-2') \quad \Gamma_1 \text{ and } \Gamma_3 \text{ in (3-2) vanish at } m.$$

This is the condition on the Γ -part of $\nabla\xi_0$ where ξ_0 is the extension to a vector field germ on M at m , which can be certainly realized.

To sum up,

(a) We extend a horizontal vector ξ_0 chosen at $z \in \mathcal{Z}$ to a vector field germ at $z \in \mathcal{Z}$ in the following way :

(a1) We first extend ξ_0 to a vector field germ on M at m so that the $\mathrm{sp}(1)$ -part of the connection form vanishes at m in the ξ_0 - and $J\xi_0$ -directions (i.e., $\alpha_1(\xi_i) = \alpha_3(\xi_i) = 0$ ($i = 0, 2$)). Here, we automatically $\alpha_2 = 0$ at m and we impose no condition on the $\mathrm{sp}(n)$ -part.

(a2) Then we extend ξ_0 as a section of the bundle $\{\mathcal{D}'_z\}_{z \in \mathbb{P}_m^1}$ in the fiber \mathbb{P}_m^1 -direction.

(b) We extend J (the orthogonal complex structure of $T_m M$ at $z \in \mathbb{P}_m^1$) to a section $\mathrm{End}^{\mathrm{skew}}(\mathcal{D})|_{\mathbb{P}_m^1}$ in the following way :

(b1) We first extend J to a section germ of $\mathcal{S} \rightarrow M$ so that $\nabla J = 0$ holds at m .

(b2) Then we extend J along the \mathbb{P}_m^1 fiber tautologically, i.e., so that $\nabla J = 2\alpha_3 \otimes I - 2\alpha_1 \otimes K$ holds.

(b3) Then we extend I and K along all directions of \mathcal{Z} so that $\nabla I = -2\alpha_3 \otimes J + 2\alpha_2 \otimes K$ and $\nabla K = -2\alpha_2 \otimes I + 2\alpha_1 \otimes J$ hold, where α_1 and α_3 are the $\mathrm{sp}(1)$ -part of the connection form, which is identified with the pulled back to \mathcal{Z} via the map germ $\mathcal{Z} \rightarrow \mathcal{P}$ constructed from the section germ (composed with the projection $\mathcal{P} \rightarrow \mathcal{Z}$) $\sigma : (M, m) \rightarrow (\mathcal{Z}, z)$ satisfying (3-1).

(b4) In §5, we will use the condition (3-2').

We would like to use the above constructed $(4n)$ -dimensional *non-horizontal* distribution \mathcal{D} on \mathcal{Z} to construct the family $\mathcal{F}^{\mathcal{Z}}$ of \mathcal{Z} -metrics on \mathcal{Z} . This attempt works most symmetric way on the canonical $\mathrm{SO}(2)$ -extension $\tilde{\mathcal{Z}}$ of \mathcal{Z} . Define the extended twistor space $\tilde{\mathcal{Z}}$ by

$$\tilde{\mathcal{Z}} = \mathcal{P}/\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{SU}(2n) .$$

The meaning of the $\mathrm{SO}(2)$ -bundle $\tilde{\mathcal{Z}} \rightarrow \mathcal{Z}$ is the following. This $\mathrm{SO}(2)$ -bundle structure defines a Hermitian holomorphic negative line bundle on the complex manifold \mathcal{Z} with its curvature form

proportional to the Kähler-Einstein form on \mathcal{Z} (see a statement just after the formula (2-8) in old version)).

We construct a family $\tilde{\mathcal{F}}^{\mathcal{Z}}$ of metrics on $\tilde{\mathcal{Z}}$ and the family $\mathcal{F}^{\mathcal{Z}}$ of Z-metrics on \mathcal{Z} is defined uniquely so that the projection $\tilde{\mathcal{Z}} \rightarrow \mathcal{Z}$ is a Riemannian submersion.

We start the construction of $\tilde{\mathcal{F}}^{\mathcal{Z}}$ by finding a basis of the 3-dimensional space spanned by α_i 's ($i = 1, 2, 3$) modified by X^j 's ($j = 0, 1, 2, 3$) which annihilate the natural lift (defined later) to $\tilde{\mathcal{Z}}$ of \mathcal{D}_z .

The $\mathfrak{sp}(1)$ -component

$$\{\alpha_1, \alpha_2, \alpha_3\}$$

of the connection form of the Levi-Civita connection of the original quaternion Kähler metric g fits into the formula

$$\nabla \begin{pmatrix} I & J & K \end{pmatrix} = \begin{pmatrix} I & J & K \end{pmatrix} \otimes \begin{pmatrix} 0 & 2\alpha_3 & -2\alpha_2 \\ -2\alpha_3 & 0 & 2\alpha_1 \\ 2\alpha_2 & -2\alpha_1 & 0 \end{pmatrix}.$$

This formula makes sense as the derivation formula of the oriented orthonormal section germ of the bundle $\mathcal{S}(\mathcal{D})$ of $\text{End}^{\text{skew}}(\mathcal{D})$ w.r.to the connection induced from the connection just as in (3-1), where the sub-bundle $\mathcal{S}(\mathcal{D}) \subset \text{End}^{\text{skew}}$ is induced from $\mathcal{S} \subset \text{End}^{\text{skew}}$ just in the same sense as (3-1). From here on, the connection ∇ which will appear in all derivation formulae should be understood in the same way. Moreover, as $\xi_1 = I\xi_0$ and so on, the above derivation formula implies

$$(3-3) \quad \nabla \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \end{pmatrix} \otimes \begin{pmatrix} 0 & 2\alpha_3 & -2\alpha_2 \\ -2\alpha_3 & 0 & 2\alpha_1 \\ 2\alpha_2 & -2\alpha_1 & 0 \end{pmatrix} \\ + \begin{pmatrix} I\nabla\xi_0 & J\nabla\xi_0 & K\nabla\xi_0 \end{pmatrix}$$

and $\nabla\xi_0$ *etc.* should be understood as

$$(3-4) \quad \begin{cases} \nabla\xi_0 = \Gamma_0\xi_0 + (\Gamma_1 + \alpha_1)\xi_1 + (\Gamma_2 + \alpha_2)\xi_2 + (\Gamma_3 + \alpha_3)\xi_3 \\ I\nabla\xi_0 = \Gamma_0\xi_1 - (\Gamma_1 + \alpha_1)\xi_0 + (\Gamma_2 + \alpha_2)\xi_3 - (\Gamma_3 + \alpha_3)\xi_2 \\ J\nabla\xi_0 = \Gamma_0\xi_2 - (\Gamma_1 + \alpha_1)\xi_3 - (\Gamma_2 + \alpha_2)\xi_0 + (\Gamma_3 + \alpha_3)\xi_1 \\ K\nabla\xi_0 = \Gamma_0\xi_3 + (\Gamma_1 + \alpha_1)\xi_2 - (\Gamma_2 + \alpha_2)\xi_1 - (\Gamma_3 + \alpha_3)\xi_0 \end{cases}.$$

Although these derivation formulae involve terms which are “invisible” at $z \in \mathcal{Z}$, we must take these terms into account, because their derivatives become significant in the curvature computation.

We introduce the system of 1-forms $\{X^0, X^1, X^2, X^3\}$ on \mathcal{Z} which should be understood as a column n vector (i.e., each X^i represents an n -dimensional subspace in the cotangent space of M) by requiring that the system $\{X^0, X^1, X^2, X^3\}$ forms a basis dual to $\{\xi_0, \xi_1 = I\xi_0, \xi_2 = J\xi_0, \xi_3 = K\xi_0\}$ in $\tilde{\mathcal{D}}$ at each point of $\tilde{\mathcal{Z}}$ and annihilates the orthogonal complement $(\tilde{\mathcal{D}}_{\tilde{z}})^\perp$ w.r.to the basic canonical deformation metric g_1^{can} on $\tilde{\mathcal{Z}}$. Of course the system of 1-forms $\{X^0, X^1, X^2, X^3\}$ is obtained from the map germ σ and the extension of $\{I, J, K\}$ in the fiber \mathbb{P}_m^1 direction by the pull-back of the canonical 1-forms on the holonomy reduction \mathcal{P} of the oriented orthonormal frames of (M, g) . Note that we can define the basic canonical deformation metric on $\tilde{\mathcal{Z}}$ just in the same way as on \mathcal{Z} . To do this we just replace the Fubini-Study metric on \mathbb{P}^1 with curvature 4 by the bi-invariant metric

on $SU(2)$ with sectional curvature identically 1. The dual version of the above formulae can be described in terms of the covariant derivative (or exterior derivative) of X^i 's

$$\begin{cases} \nabla \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix} + \begin{pmatrix} 0 & 2\alpha_3 & -2\alpha_2 \\ -2\alpha_3 & 0 & 2\alpha_1 \\ 2\alpha_2 & -2\alpha_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix} + \begin{pmatrix} I\nabla X^0 \\ J\nabla X^0 \\ K\nabla X^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \nabla X^0 = -\Gamma_0 \otimes X^0 + (\Gamma_1 + \alpha_1) \otimes X^1 + (\Gamma_2 + \alpha_2) \otimes X^2 + (\Gamma_3 + \alpha_3) \otimes X^3 \end{cases}$$

or equivalently

$$(3-5) \quad \begin{cases} dX^1 - 2\alpha_2 \wedge X^3 + 2\alpha_3 \wedge X^2 + IdX^0 = 0, \\ dX^2 - 2\alpha_3 \wedge X^1 + 2\alpha_1 \wedge X^3 + JdX^0 = 0, \\ dX^3 - 2\alpha_1 \wedge X^2 + 2\alpha_2 \wedge X^1 + KdX^0 = 0, \\ dX^0 = -\Gamma_0 \wedge X^0 + (\Gamma_1 + \alpha_1) \wedge X^1 + (\Gamma_2 + \alpha_2) \wedge X^2 + (\Gamma_3 + \alpha_3) \wedge X^3 \end{cases}$$

(all coefficients are “invisible” at $z \in \mathcal{Z}$). These formulae hold on \mathcal{P} before the projection onto the (extended) twistor space. Therefore we can speak of the Sasakian structure on $\tilde{\mathcal{Z}}$ and its transversal Kähler structure. The merit of considering $\tilde{\mathcal{Z}}$ is that we have a canonical choice of the triple $\{I, J, K\}$ ($J = J(z)$, $z \in \mathcal{Z}$) along the $SO(2)$ -fiber over z , while in the above construction on \mathcal{Z} the choice of $\{I, K\}$ is determined only modulo $SO(2)$ -rotation. Once ξ_0 is chosen as in the above discussion, we get an orthonormal frame germ $\{\xi_0, I\xi_0, J\xi_0, K\xi_0\}$ at m . By this procedure we have defined a (germ of) section of the extended twistor bundle $\tilde{\mathcal{Z}} \rightarrow M$ which is horizontal in the $\{\xi_0, J\xi_0\}_{\text{span}}$ -direction but *not* horizontal in the $\{I\xi_0, K\xi_0\}_{\text{span}}$ -direction. We have thus defined a $(4n)$ -dimensional distribution $\tilde{\mathcal{D}}$ on $\tilde{\mathcal{Z}}$ which inherits the same property as \mathcal{D} originally defined on \mathcal{Z} .

We are now ready to define the Z-metric on the extended twistor space $\tilde{\mathcal{Z}}$. We note that the system of 1-form germs

$$\begin{cases} \alpha_1 - \alpha_1(\xi_0)X^0 - \alpha_1(\xi_1)X^1 - \alpha_1(\xi_2)X^2 - \alpha_1(\xi_3)X^3 \\ \alpha_2 - \alpha_2(\xi_0)X^0 - \alpha_2(\xi_1)X^1 - \alpha_2(\xi_2)X^2 - \alpha_2(\xi_3)X^3 \\ \alpha_3 - \alpha_3(\xi_0)X^0 - \alpha_3(\xi_1)X^1 - \alpha_3(\xi_2)X^2 - \alpha_3(\xi_3)X^3 \end{cases}$$

at $\tilde{z} \in \tilde{\mathcal{Z}}$ (respectively

$$\begin{cases} \alpha_1 - \alpha_1(\xi_0)X^0 - \alpha_1(\xi_1)X^1 - \alpha_1(\xi_2)X^2 - \alpha_1(\xi_3)X^3 \\ \alpha_3 - \alpha_3(\xi_0)X^0 - \alpha_3(\xi_1)X^1 - \alpha_3(\xi_2)X^2 - \alpha_3(\xi_3)X^3 \end{cases}$$

at $z \in \mathcal{Z}$) annihilates the $(4n)$ -dimensional distribution $\tilde{\mathcal{D}}$ on $\tilde{\mathcal{Z}}$ (resp. \mathcal{D} on \mathcal{Z}). However, these 1-form germs are not mutually orthogonal at $\tilde{z} \in \tilde{\mathcal{Z}}$ (resp. at $z \in \mathcal{Z}$) w.r.to the canonical deformation metrics. Indeed, although we have $\alpha_1(\xi_i) = \alpha_2(\xi_i) = \alpha_3(\xi_i) = 0$ ($i = 0, 2$) at $\tilde{z} \in \tilde{\mathcal{Z}}$ and $\alpha_2 = 0$ for every direction at $z \in \mathcal{Z}$, we have $\alpha_i(\xi_j) \neq 0$ for $i, j = 1, 3$ (resp. for $i = 1, 2, 3$ and $j = 1, 3$) at $z \in \mathcal{Z}$ (resp. $\tilde{z} \in \tilde{\mathcal{Z}}$) which make these 1-forms mutually non orthogonal w.r.to g_λ^{can} . Here we should write ${}^t\alpha_k(\xi_i)X^i$ instead of $\alpha_k(\xi_i)X^i$ because $\alpha_k(\xi_i)$ should be interpreted as a row vector which makes “inner product” with a column vector X^i . However, for the brevity, we have omit

the transposition sign. We define the family of Z-metrics $\tilde{\mathcal{F}}^Z$ on $\tilde{\mathcal{Z}}$ in the following way. For the definition, we use the basic canonical deformation metric g_1^{can} . Before writing down the explicit form of the Z-metrics, we must check the effect that the vectors ξ_1 and ξ_3 are not horizontal. We recall that both ξ_1 and ξ_3 decompose into the orthogonal sum of column $(n-1)$ -vector in \mathcal{D}'_z (interpreted as an $(n-1)$ -dimensional subspace) and a unit vector in $L'_{J(z)}$. The length of these components w.r.to the basic canonical deformation metric g_1^{can} are $n-1$ and $\sqrt{2}$ respectively. Therefore the quadruple

$$\left\{ \xi_0, \sqrt{\frac{n}{n+1}}\xi_1, \xi_2, \sqrt{\frac{n}{n+1}}\xi_3 \right\}$$

is an orthonormal basis of \mathcal{D}_z w.r.to the basic canonical deformation metric. We introduce normalized version of ξ_i ($i = 1, 3$) and X^i ($i = 1, 3$) by putting

$$\xi'_1 := \sqrt{\frac{n}{n+1}}\xi_1, \quad \xi'_3 := \sqrt{\frac{n}{n+1}}\xi_3, \quad X^1_n := \sqrt{\frac{n+1}{n}}X^1, \quad X^3_n := \sqrt{\frac{n+1}{n}}X^3.$$

Then $\{\xi_0, \xi'_1, \xi_2, \xi'_3\}$ is an orthonormal basis of \mathcal{D}_z w.r.to the basic canonical deformation metric g_1^{can} and the quadruple $\{X^0, X^1_n, X^2, X^3_n\}$ is the dual basis of $(\mathcal{D}'_z)^*$. We extend these 1-forms so that these vanish on vectors in $(\mathcal{D}_z)^\perp$ where the orthogonal complement \perp is taken w.r.to the basic canonical deformation metric g_1^{can} . We thus have a frame germ

$$\left\{ \lambda(\alpha_1 - \sum_{i=0}^3 \alpha_1(\xi_i)X^i), \lambda(\alpha_2 - \sum_{i=0}^3 \alpha_2(\xi_i)X^i), \lambda(\alpha_3 - \sum_{i=0}^3 \alpha_3(\xi_i)X^i), X^0, X^1_n, X^2, X^3_n \right\}.$$

One would like to define a family of metrics on $\tilde{\mathcal{Z}}$ by declaring that the above frame germ (after applying some orthogonalization procedure to the first three 1-forms w.r.to the basic canonical deformation metric) being orthonormal coframe (modulo total scaling parameter). However, this attempt of defining new metrics does not fit with our strategy of constructing new metrics on \mathcal{Z} (or $\tilde{\mathcal{Z}}$) which is “closer” to the independently scaled product metrics of $S^n \times S^n$. Instead, we define the family of Z-metrics by declaring that the frame germ

$$\left\{ \lambda(\alpha_1 - \sum_{i=0}^3 \alpha_1(\xi_i)X^i), \lambda(\alpha_2 - \sum_{i=0}^3 \alpha_2(\xi_i)X^i), \lambda(\alpha_3 - \sum_{i=0}^3 \alpha_3(\xi_i)X^i), X^0, X^1, X^2, X^3 \right\}.$$

(without normalization on X^1 and X^3) being orthonormal (modulo total scaling parameter ρ) :

$$\begin{aligned} \rho \tilde{g}^Z_\lambda := & \rho \left[\lambda^2 \left\{ \left(\alpha_1 - \alpha_1(\xi_0)X^0 - \alpha_1(\xi_1)X^1 - \alpha_1(\xi_2)X^2 - \alpha_1(\xi_3)X^3 \right)^2 \right. \right. \\ & + \left(\alpha_2 - \alpha_2(\xi_0)X^0 - \alpha_2(\xi_1)X^1 - \alpha_2(\xi_2)X^2 - \alpha_2(\xi_3)X^3 \right)^2 \\ & + \left. \left(\alpha_3 - \alpha_3(\xi_0)X^0 - \alpha_3(\xi_1)X^1 - \alpha_3(\xi_2)X^2 - \alpha_3(\xi_3)X^3 \right)^2 \right\} \\ & + {}^tX^0 \cdot X^0 + {}^tX^1 \cdot X^1 + {}^tX^2 \cdot X^2 + {}^tX^3 \cdot X^3 \Big]. \end{aligned}$$

The family \mathcal{F}^Z of Z-metrics on the twistor space \mathcal{Z} is defined by declaring that the $\text{SO}(2)$ -bundle $\tilde{\mathcal{Z}} \rightarrow \mathcal{Z}$ is a Riemannian submersion w.r.to Z-metrics on $\tilde{\mathcal{Z}}$ and \mathcal{Z} :

$$\begin{aligned} \rho g_\lambda^Z := \rho \left[\lambda^2 \left\{ \left(\alpha_1 - \alpha_1(\xi_0)X^0 - \alpha_1(\xi_1)X^1 - \alpha_1(\xi_2)X^2 - \alpha_1(\xi_3)X^3 \right)^2 \right. \right. \\ \left. \left. + \left(\alpha_3 - \alpha_3(\xi_0)X^0 - \alpha_3(\xi_1)X^1 - \alpha_3(\xi_2)X^2 - \alpha_3(\xi_3)X^3 \right)^2 \right\} \right. \\ \left. + {}^tX^0 \cdot X^0 + {}^tX^1 \cdot X^1 + {}^tX^2 \cdot X^2 + {}^tX^3 \cdot X^3 \right]. \end{aligned}$$

Here λ is a positive partial scaling parameter. This expression should be understood as an expression in terms of 1-form germs at $z \in \mathcal{Z}$ under consideration. These expressions define Z-metrics by specifying the oriented orthonormal coframe at one point $\tilde{z} \in \tilde{\mathcal{Z}}$ and $z \in \mathcal{Z}$. Moreover, we regard these as metric germs at $\tilde{z} \in \tilde{\mathcal{Z}}$ or $z \in \mathcal{Z}$ and compute their curvature form (we included “invisible” terms in the definition of the Z-metrics because these terms are significant in the computation of the Levi-Civita connection and the curvature form by differentiation). Even if the above expression is in the nice form only at one point (z and \tilde{z}) under consideration, the curvature computation regarding these metrics as germs is justified (we show this later). In these expressions, we note that the meaning of X^i 's ($i = 0, 1, 2, 3$) in the definition of the Z-metrics is *not the same* as that in the definition of the canonical deformation metrics. In the definition of the canonical deformation metrics, the corresponding orthonormal basis consists of vertical and horizontal vectors, while in the definition of Z-metrics the corresponding orthonormal basis does not consist of vertical / horizontal vectors. Indeed, $\alpha_1 - \alpha_1(\xi_1)X^1 - \alpha_1(\xi_3)X^3$ and $\alpha_3 - \alpha_3(\xi_1)X^1 - \alpha_3(\xi_3)X^3$ (resp. ξ_1 and ξ_3) are not vertical (resp. not horizontal)⁹, where we ignored “invisible” parts at the point under consideration.

We set

$$\begin{cases} \hat{\alpha}_1 := \alpha_1 - \alpha_1(\xi_0)X^0 - \alpha_1(\xi_1)X^1 - \alpha_1(\xi_2)X^2 - \alpha_1(\xi_3)X^3, \\ \hat{\alpha}_3 := \alpha_3 - \alpha_3(\xi_0)X^0 - \alpha_3(\xi_1)X^1 - \alpha_3(\xi_2)X^2 - \alpha_3(\xi_3)X^3. \end{cases}$$

These are regarded as 1-form germs at a point $z \in \mathcal{Z}$ under consideration. We have from (2-12) the formulae of covariant derivatives w.r.to the Levi-Civita connection of the basic canonical deformation metric g_1^{can} :

$$(3-6) \quad \begin{cases} \nabla \alpha_1 = 2\alpha_2 \otimes \alpha_3 - {}^tX^0 \otimes X^1 + {}^tX^1 \otimes X^0 + {}^tX^2 \otimes X^3 - {}^tX^3 \otimes X^2, \\ \nabla \alpha_3 = -2\alpha_2 \otimes \alpha_1 - {}^tX^0 \otimes X^3 + {}^tX^3 \otimes X^0 + {}^tX^1 \otimes X^2 - {}^tX^2 \otimes X^1, \end{cases}$$

where ∇ is the Levi-Civita connection of g and the convention is that the Levi-Civita connection form appears like (connection form \otimes section) in the above formulae (we just replace \otimes by \wedge to get (2-12) from the above covariant derivative formula in this convention).

We prepare some useful formulae. Applying K and I to the moving frame formula for dX^0 (note that $IX^0 = -X^1$, $JX^0 = -X^2$, $KX^0 = -X^3$ and so on), we have

$$(3-7) \quad \begin{cases} dX^0 = -\Gamma_0 \wedge X^0 + (\Gamma_1 + \alpha_1) \wedge X^1 + (\Gamma_2 + \alpha_2) \wedge X^2 + (\Gamma_3 + \alpha_3) \wedge X^3, \\ IdX^0 = (\Gamma_1 + \alpha_1) \wedge X^0 + \Gamma_0 \wedge X^1 + (\Gamma_3 + \alpha_3) \wedge X^2 - (\Gamma_2 + \alpha_2) \wedge X^3, \\ JdX^0 = (\Gamma_2 + \alpha_2) \wedge X^0 - (\Gamma_3 + \alpha_3) \wedge X^1 + \Gamma_0 \wedge X^2 + (\Gamma_1 + \alpha_1) \wedge X^3, \\ KdX^0 = (\Gamma_3 + \alpha_3) \wedge X^0 + (\Gamma_2 + \alpha_2) \wedge X^1 - (\Gamma_1 + \alpha_1) \wedge X^2 + \Gamma_0 \wedge X^3. \end{cases}$$

⁹ However, X^i 's ($i = 0, 1, 2, 3$) have the *same* meaning in the sense that in either cases they stem from the canonical 1-forms defined on \mathcal{P} .

From (3-4), we have ($i = 1, 3$):

$$(3-8) \quad \begin{cases} \alpha_i(\nabla \xi_0) = \alpha_i(\xi_0)\Gamma_0 + \alpha_i(\xi_1)(\Gamma_1 + \alpha_1) + \alpha_i(\xi_2)(\Gamma_2 + \alpha_2) + \alpha_i(\xi_3)(\Gamma_3 + \alpha_3) , \\ \alpha_i(I\nabla \xi_0) = \alpha_i(\xi_1)\Gamma_0 - \alpha_i(\xi_0)(\Gamma_1 + \alpha_1) + \alpha_i(\xi_3)(\Gamma_2 + \alpha_2) - \alpha_i(\xi_2)(\Gamma_3 + \alpha_3) , \\ \alpha_i(J\nabla \xi_0) = \alpha_i(\xi_2)\Gamma_0 - \alpha_i(\xi_3)(\Gamma_1 + \alpha_1) - \alpha_i(\xi_0)(\Gamma_2 + \alpha_2) + \alpha_i(\xi_1)(\Gamma_3 + \alpha_3) , \\ \alpha_i(K\nabla \xi_0) = \alpha_i(\xi_3)\Gamma_0 + \alpha_i(\xi_2)(\Gamma_1 + \alpha_1) - \alpha_i(\xi_1)(\Gamma_2 + \alpha_2) - \alpha_i(\xi_0)(\Gamma_3 + \alpha_3) . \end{cases}$$

We recall that Γ_0 and Γ_2 are not “invisible” in the ξ_1 - and ξ_3 - directions and moreover Γ_1 is invisible in the ξ_0 - and ξ_2 -directions but not invisible in the ξ_1 and ξ_2 -directions. From (3-7) and (3-8), we have, modulo terms vanishing to order ≥ 2 at $z \in \mathcal{Z}$, the following formulae:

$$(3-9) \quad \begin{aligned} & \alpha_3(\xi_1)IdX^0 + \alpha_3(\xi_3)KdX^0 \\ &= \alpha_3(\nabla \xi_0) \wedge X^0 + \alpha_3(I\nabla \xi_0) \wedge X^1 + \alpha_3(J\nabla \xi_0) \wedge X^2 + \alpha_3(K\nabla \xi_0) \wedge X^3 \\ & \quad + \{\alpha_3(\xi_0)(\Gamma_1 + \alpha_1) + \alpha_3(\xi_2)(\Gamma_3 + \alpha_3)\} \wedge X^1 \\ & \quad + \{-\alpha_3(\xi_2)(\Gamma_1 + \alpha_1) + \alpha_3(\xi_0)(\Gamma_3 + \alpha_3)\} \wedge X^3 \\ & \quad - \alpha_3(\xi_0)\Gamma_0 \wedge X^0 - \alpha_3(\xi_2)\Gamma_2 \wedge X^0 - \alpha_3(\xi_2)\Gamma_0 \wedge X^2 - \alpha_3(\xi_0)\Gamma_2 \wedge X^2 , \\ & \alpha_1(\xi_3)KdX^0 + \alpha_1(\xi_1)IdX^0 \\ &= \alpha_1(\nabla \xi_0) \wedge X^0 + \alpha_1(I\nabla \xi_0) \wedge X^1 + \alpha_1(J\nabla \xi_0) \wedge X^2 + \alpha_1(K\nabla \xi_0) \wedge X^3 \\ & \quad + \{\alpha_1(\xi_0)(\Gamma_1 + \alpha_1) + \alpha_1(\xi_2)(\Gamma_3 + \alpha_3)\} \wedge X^1 \\ & \quad + \{-\alpha_1(\xi_2)(\Gamma_1 + \alpha_1) + \alpha_1(\xi_0)(\Gamma_3 + \alpha_3)\} \wedge X^3 \\ & \quad - \alpha_1(\xi_0)\Gamma_0 \wedge X^0 - \alpha_1(\xi_2)\Gamma_2 \wedge X^0 - \alpha_1(\xi_2)\Gamma_0 \wedge X^2 - \alpha_1(\xi_0)\Gamma_2 \wedge X^2 . \end{aligned}$$

Moreover we have

$$(3-10) \quad \begin{aligned} dX^0 &= (\Gamma_1 + \alpha_1) \wedge X^1 + (\Gamma_3 + \alpha_3) \wedge X^3 , \\ IdX^0 &= (\Gamma_1 + \alpha_1) \wedge X^0 + (\Gamma_3 + \alpha_3) \wedge X^2 , \\ JdX^0 &= -(\Gamma_3 + \alpha_3) \wedge X^1 + (\Gamma_1 + \alpha_1) \wedge X^3 , \\ KdX^0 &= (\Gamma_3 + \alpha_3) \wedge X^0 - (\Gamma_1 + \alpha_1) \wedge X^2 \end{aligned}$$

at $z \in \mathcal{Z}$ modulo “invisible” terms. Now we compute the derivation formulae for $\hat{\alpha}_1$ and $\hat{\alpha}_3$ modulo terms vanishing to order ≥ 2 at $z \in \mathcal{Z}$. We have

$$\begin{aligned} d\hat{\alpha}_1 &= d\left(\alpha_1 - \sum_{i=9}^3 \alpha_1(\xi_i)X^i\right) \\ &= d\alpha_1 - \sum_{i=0}^3 \left\{ (\nabla \alpha_1)(\xi_i) \wedge X^i + \alpha_1(\nabla \xi_i) \wedge X^i + \alpha_1(\xi_i)dX^i \right\} \end{aligned}$$

where ∇ is the Levi-Civita connection of the basic canonical deformation metric g_1^{can} . Applying

(2-12), (3-6), (3-3) and (3-5) to $d\alpha_1$, $\nabla\alpha_1$, $\nabla\xi_i$ and dX^i , we compute the right hand side as

$$\begin{aligned}
& 2\alpha_2 \wedge \alpha_3 + 2({}^tX^1 \wedge X^0 + {}^tX^2 \wedge X^3) \\
& - (2\alpha_2 \otimes \alpha_3 - {}^tX^0 \otimes X^1 + {}^tX^1 \otimes X^0 + {}^tX^2 \otimes X^3 - {}^tX^3 \otimes X^2)(\xi_0) \wedge X^0 \\
& - \alpha_1(\nabla\xi_0) \wedge X^0 \\
& - \alpha_1(\xi_0)dX^0 \\
& - (2\alpha_2 \otimes \alpha_3 - {}^tX^0 \otimes X^1 + {}^tX^1 \otimes X^0 + {}^tX^2 \otimes X^3 - {}^tX^3 \otimes X^2)(\xi_1) \wedge X^1 \\
& - \alpha_1(2\alpha_2 \otimes \xi_3 - 2\alpha_3 \otimes \xi_2 + I\nabla\xi_0) \wedge X^1 \\
& - \alpha_1(\xi_1)(2\alpha_2 \wedge X^3 - 2\alpha_3 \wedge X^2 - IdX^0) \\
& - 2(2\alpha_2 \otimes \alpha_3 - {}^tX^0 \otimes X^1 + {}^tX^1 \otimes X^0 + {}^tX^2 \otimes X^3 - {}^tX^3 \otimes X^2)(\xi_2) \wedge X^2 \\
& - \alpha_1(2\alpha_3 \otimes \xi_1 - 2\alpha_1 \otimes \xi_3 + J\nabla\xi_0) \wedge X^2 \\
& - \alpha_1(\xi_2)(2\alpha_3 \wedge X^1 - 2\alpha_1 \wedge X^3 - JdX^0) \\
& - (2\alpha_2 \otimes \alpha_3 - {}^tX^0 \otimes X^1 + {}^tX^1 \otimes X^0 + {}^tX^2 \otimes X^3 - {}^tX^3 \otimes X^2)(\xi_3) \wedge X^3 \\
& - \alpha_1(2\alpha_1 \otimes \xi_2 - 2\alpha_2 \otimes \xi_1 + K\nabla\xi_0) \wedge X^3 \\
& - \alpha_1(\xi_3)(2\alpha_1 \wedge X^2 - 2\alpha_2 \wedge X^1 - KdX^0) .
\end{aligned}$$

Executing the cancellation and applying (18), we can reduce this, modulo terms vanishing at $z \in \mathcal{Z}$ to order ≥ 2 , to the following :

$$\begin{aligned}
& 2\alpha_2 \wedge \alpha_3 - 2\alpha_3(\xi_0)\alpha_2 \wedge X^0 - 2\alpha_3(\xi_1)\alpha_2 \wedge X^1 - 2\alpha_3(\xi_2)\alpha_2 \wedge X^2 - \alpha_3(\xi_3)\alpha_2 \wedge X^3 \\
& - \alpha_1(\xi_0)\{(\Gamma_1 + \alpha_1) \wedge X^1 + (\Gamma_3 + \alpha_3) \wedge X^3\} \\
& + \alpha_1(\xi_2)\{-(\Gamma_3 + \alpha_3) \wedge X^1 + (\Gamma_1 + \alpha_1) \wedge X^3\} \\
& - \alpha_1(\nabla\xi_0) \wedge X^0 - \alpha_1(I\nabla\xi_0) \wedge X^1 - \alpha_1(J\nabla\xi_0) \wedge X^2 - \alpha_1(K\nabla\xi_0) \wedge X^3 \\
& + \alpha_1(\xi_3)KdX^0 + \alpha_1(\xi_1)IdX^0 .
\end{aligned}$$

Using the definition of $\hat{\alpha}_3$ and applying (17), this becomes

$$\begin{aligned}
& 2\alpha_2 \wedge \hat{\alpha}_3 - \alpha_1(\xi_0)\{(\Gamma_1 + \alpha_1) \wedge X^1 + (\Gamma_3 + \alpha_3) \wedge X^3\} \\
& + \alpha_1(\xi_2)\{-(\Gamma_3 + \alpha_3) \wedge X^1 + (\Gamma_1 + \alpha_1) \wedge X^3\} \\
& + \{\alpha_1(\xi_0)(\Gamma_1 + \alpha_1) + \alpha_1(\xi_2)(\Gamma_3 + \alpha_3)\} \wedge X^1 \\
& + \{-\alpha_1(\xi_2)(\Gamma_1 + \alpha_1) + \alpha_1(\xi_0)(\Gamma_3 + \alpha_3)\} \wedge X^3 \\
& - \alpha_1(\xi_0)\Gamma_0 \wedge X^0 - \alpha_1(\xi_2)\Gamma_2 \wedge X^0 - \alpha_1(\xi_2)\Gamma_0 \wedge X^2 - \alpha_1(\xi_0)\Gamma_2 \wedge X^2
\end{aligned}$$

and finally we get the derivation formula for $\hat{\alpha}_1$ (modulo terms vanishing at z to order ≥ 2):

(3-11)

$$\begin{aligned}
d\hat{\alpha}_1 &= 2\alpha_2 \wedge \hat{\alpha}_3 \\
& - \alpha_1(\xi_0)\Gamma_0 \wedge X^0 - \alpha_1(\xi_2)\Gamma_2 \wedge X^0 - \alpha_1(\xi_2)\Gamma_0 \wedge X^2 - \alpha_1(\xi_0)\Gamma_2 \wedge X^2 .
\end{aligned}$$

Next we compute $d\hat{\alpha}_3$. Computing just as in the same way, we have

$$\begin{aligned}
d\hat{\alpha}_3 &= d\left(\alpha_3 - \sum_{i=0}^3 \alpha_3(\xi_i)X^i\right) \\
&= d\alpha_3 - \sum_{i=0}^3 \left\{ (\nabla\alpha_3)(\xi_i)X^i + \alpha_3(\nabla\xi_i)X^i + \alpha_3(\xi_i)dX^i \right\} \\
&= -2\alpha_2 \wedge \alpha_1 + 2({}^tX^3 \wedge X^0 + {}^tX^1 \wedge X^2) \\
&\quad - (-2\alpha_2 \otimes \alpha_1 - {}^tX^0 \otimes X^3 + {}^tX^3 \otimes X^0 + {}^tX^1 \otimes X^2 - {}^tX^2 \otimes X^1)(\xi_0) \wedge X^0 \\
&\quad - \alpha_3(\nabla\xi_0) \wedge X^0 \\
&\quad - \alpha_3(\xi_0)dX^0 \\
&\quad - (-2\alpha_2 \otimes \alpha_1 - {}^tX^0 \otimes X^3 + {}^tX^3 \otimes X^0 + {}^tX^1 \otimes X^2 - {}^tX^2 \otimes X^1)(\xi_1) \wedge X^1 \\
&\quad - \alpha_3(2\alpha_2 \otimes \xi_3 - 2\alpha_3 \otimes \xi_2 + I\nabla\xi_0) \wedge X^1 \\
&\quad - \alpha_3(\xi_1)(2\alpha_2 \wedge X^3 - 2\alpha_3 \wedge X^2 - IdX^0) \\
&\quad - 2(-2\alpha_2 \otimes \alpha_1 - {}^tX^0 \otimes X^3 + {}^tX^3 \otimes X^0 + {}^tX^1 \otimes X^2 - {}^tX^2 \otimes X^1)(\xi_2) \wedge X^2 \\
&\quad - \alpha_3(2\alpha_3 \otimes \xi_1 - 2\alpha_1 \otimes \xi_3 + J\nabla\xi_0) \wedge X^2 \\
&\quad - \alpha_3(\xi_2)(2\alpha_3 \wedge X^1 - 2\alpha_1 \wedge X^3 - JdX^0) \\
&\quad - (-2\alpha_2 \otimes \alpha_1 - {}^tX^0 \otimes X^3 + {}^tX^3 \otimes X^0 + {}^tX^1 \otimes X^2 - {}^tX^2 \otimes X^1)(\xi_3) \wedge X^3 \\
&\quad - \alpha_3(2\alpha_1 \otimes \xi_2 - 2\alpha_2 \otimes \xi_1 + K\nabla\xi_0) \\
&\quad - \alpha_3(\xi_3)(-2\alpha_1 \wedge X^2 + 2\alpha_2 \wedge X^1 - KdX^0) .
\end{aligned}$$

Executing cancellation and using (18), the right hand side reduces to

$$\begin{aligned}
&-2\alpha_2 \wedge \alpha_1 + 2\alpha_1(\xi_0)\alpha_2 \wedge X^0 + 2\alpha_1(\xi_1)\alpha_2 \wedge X^1 + 2\alpha_1(\xi_2)\alpha_2 \wedge X^2 + \alpha_1(\xi_3)\alpha_2 \wedge X^3 \\
&- \alpha_3(\xi_0)\{(\Gamma_1 + \alpha_1) \wedge X^1 + (\Gamma_3 + \alpha_3) \wedge X^3\} \\
&+ \alpha_3(\xi_2)\{-(\Gamma_3 + \alpha_3) \wedge X^1 + (\Gamma_1 + \alpha_1) \wedge X^3\} \\
&- \alpha_3(\nabla\xi_0) \wedge X^0 - \alpha_3(I\nabla\xi_0) \wedge X^1 - \alpha_3(J\nabla\xi_0) \wedge X^2 - \alpha_3(K\nabla\xi_0) \wedge X^3 \\
&+ \alpha_3(\xi_1)IdX^0 + \alpha_3(\xi_3)KdX^0 .
\end{aligned}$$

We compute just in the same way as above using the definition of $\hat{\alpha}_1$ and applying (3-9), (3-10) to get the derivation formula of $\hat{\alpha}_3$ (modulo terms vanishing at z to order ≥ 2):

(3-12)

$$\begin{aligned}
d\hat{\alpha}_3 &= -2\alpha_2 \wedge \hat{\alpha}_1 - \alpha_3(\xi_0)\{(\Gamma_1 + \alpha_1) \wedge X^1 + (\Gamma_3 + \alpha_3) \wedge X^3\} \\
&\quad + \alpha_3(\xi_2)\{-(\Gamma_3 + \alpha_3) \wedge X^1 + (\Gamma_1 + \alpha_1) \wedge X^3\} \\
&\quad + \{\alpha_3(\xi_0)(\Gamma_1 + \alpha_1) + \alpha_3(\xi_2)(\Gamma_3 + \alpha_3)\} \wedge X^1 \\
&\quad + \{-\alpha_3(\xi_2)(\Gamma_1 + \alpha_1) + \alpha_3(\xi_0)(\Gamma_3 + \alpha_3)\} \wedge X^3 \\
&\quad - \alpha_3(\xi_0)\Gamma_0 \wedge X^0 - \alpha_3(\xi_2)\Gamma_2 \wedge X^0 - \alpha_3(\xi_2)\Gamma_0 \wedge X^2 - \alpha_3(\xi_0)\Gamma_2 \wedge X^2 \\
&= -2\alpha_2 \wedge \hat{\alpha}_1 \\
&\quad - \alpha_3(\xi_0)\Gamma_0 \wedge X^0 - \alpha_3(\xi_2)\Gamma_2 \wedge X^0 - \alpha_3(\xi_2)\Gamma_0 \wedge X^2 - \alpha_3(\xi_0)\Gamma_2 \wedge X^2 .
\end{aligned}$$

Let us compare the derivation formulae (3-11) and (3-12) of $\hat{\alpha}_i$ ($i = 1, 3$) with those (2-12) of α_i ($i = 1, 3$). The remarkable difference is the following. In (2-12), the term such as $X^\mu \wedge X^\nu$ appears (geometrically this means that the curvature form of the canonical Hermitian metric of the S^1 -bundle $\tilde{\mathcal{Z}} \rightarrow \mathcal{Z}$ is proportional to the Kähler-Einstein metric g_1^{can} on \mathcal{Z}). On the other hand, although there are similar terms (such as $-\alpha_3(\xi_0)\Gamma_0 \wedge X^0$) in (3-11) and (3-12), these terms appear with coefficients “invisible” at $z\mathcal{Z}$ (such as $\alpha_3(\xi_0)$). We are thus tempted to think that the distribution \mathcal{D} on the twistor space \mathcal{Z} defined by the equation $\hat{\alpha}_1 = \hat{\alpha}_3 = 0$ is much “closer to being integrable” than is the horizontal distribution \mathcal{H} of the twistor fibration $\mathcal{Z} \rightarrow M$ (w.r.to the Levi-Civita connection of the quaternion Kähler metric g on M). In other words, we are tempted to think that the distribution $\mathcal{D} = \{\hat{\alpha}_1 = \hat{\alpha}_3 = 0\}$ on the twistor space \mathcal{Z} equipped with the Z-metrics g_λ^Z constructed from the distribution \mathcal{D} would be much “closer” to the product structure on, e.g., $S^n \times S^n$ equipped with the independently scaled product constant curvature metrics than is the horizontal distribution \mathcal{H} equipped with the canonical deformation metrics.

We are now ready to compute the Levi-Civita connection of the Z-metrics. We recall that the Γ -part of the Levi-Civita connection of g satisfies the condition that Γ_0 and Γ_2 are “invisible” in ξ_0 - and $J\xi_0$ -directions (this stems from the construction of the distribution \mathcal{D} on \mathcal{Z}). This implies that we can write

$$\begin{aligned}\Gamma_0 &= p X^1 + q X^3, \\ \Gamma_2 &= r X^1 + s X^3.\end{aligned}$$

Here, these expressions should be understood in the column n vector notation under the identification $\mathbb{R}^{4n} = \mathbb{H}^n$. We write down the first structure equation of the metric g_λ^Z on \mathcal{Z} w.r.to the moving frame

$$\{\lambda\hat{\alpha}_1, \lambda\hat{\alpha}_3, X^0, X^1, X^2, X^3\}$$

at $z \in \mathcal{Z}$. We recall that Z-metric g_λ^Z is defined by declaring that the above moving frame is an oriented orthonormal frame. We write this as a column vector as in the first structure equation in §2, i.e., the connection matrix Γ_λ^Z is defined by

$$d \begin{pmatrix} \lambda\hat{\alpha}_1 \\ \lambda\hat{\alpha}_3 \\ X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} + \Gamma_\lambda^Z \wedge \begin{pmatrix} \lambda\hat{\alpha}_1 \\ \lambda\hat{\alpha}_3 \\ X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For this purpose, we put, for $i = 1, 3$:

$$\begin{aligned}a_i &:= \frac{1}{2} \{ \alpha_i(\xi_0)p + \alpha_i(\xi_2)r \}, \\ b_i &:= \frac{1}{2} \{ \alpha_i(\xi_0)q + \alpha_i(\xi_2)s \}, \\ c_i &:= \frac{1}{2} \{ \alpha_i(\xi_2)p + \alpha_i(\xi_0)r \}, \\ d_i &:= \frac{1}{2} \{ \alpha_i(\xi_2)q + \alpha_i(\xi_0)s \}.\end{aligned}$$

Note that these are all “invisible” at $z \in \mathcal{Z}$. It follows from the derivation formulae (3-11) and (3-12) that the connection matrix Γ_λ^Z under question is written as

$$(3-13) \quad \begin{pmatrix} 0 & -2\alpha_2 & \lambda(a_1X^1 + b_1X^3) & -\lambda(a_1X^0 + c_1X^2) & \lambda(c_1X^1 + d_1X^3) & -\lambda(b_1X^0 + d_1X^2) \\ 2\alpha_2 & 0 & \lambda((a_3X^1 + b_3X^3) & -\lambda(a_3X^0 + c_3X^2) & \lambda(c_3X^1 + d_3X^3) & -\lambda(b_3X^0 + d_3X^2) \\ & & -\Gamma_1 - \alpha_1 & & & -\Gamma_3 - \alpha_3 \\ & & -\lambda^2(a_1\alpha_1 + a_3\alpha_3) & & & -\lambda^2(b_1\alpha_1 + b_3\alpha_3) \\ -\lambda(a_1X^1 & -\lambda(a_3X^1 & \Gamma_0 & -a_1\alpha_1(\xi_1)X^1 & -\Gamma_2 + \alpha_2 & -b_1\alpha_1(\xi_1)X^1 \\ +b_1X^3) & +b_3X^3) & +0 & -a_1\alpha_1(\xi_3)X^3 & +0 & -b_1\alpha_1(\xi_3)X^3 \\ & & & -a_3\alpha_3(\xi_1)X^1 & & -b_3\alpha_3(\xi_1)X^1 \\ & & & -a_3\alpha_3(\xi_3)X^3) & & -b_3\alpha_3(\xi_3)X^3) \\ & & \Gamma_1 + \alpha_1 & & -\Gamma_3 + \alpha_3 & \\ & & +\lambda^2(a_1\alpha_1 + a_3\alpha_3) & & +\lambda^2(c_1\alpha_1 + c_3\alpha_3) & \\ \lambda(a_1X^0 & \lambda(a_3X^0 & -a_1\alpha_1(\xi_1)X^1 & \Gamma_0 & -c_1\alpha_1(\xi_1)X^1 & \Gamma_2 - \alpha_2 \\ +c_1X^2) & +c_3X^2) & -a_1\alpha_1(\xi_3)X^3 & +0 & -c_1\alpha_1(\xi_3)X^3 & +0 \\ & & -a_3\alpha_3(\xi_1)X^1 & & -c_3\alpha_3(\xi_1)X^1 & \\ & & -a_3\alpha_3(\xi_3)X^3) & & -c_3\alpha_3(\xi_3)X^3) & \\ & & \Gamma_3 - \alpha_3 & & -\Gamma_1 + \alpha_1 & \\ & & -\lambda^2(c_1\alpha_1 + c_3\alpha_3) & & -\lambda^2(d_1\alpha_1 + d_3\alpha_3) & \\ -\lambda(c_1X^1 & -\lambda(c_3X^1 & \Gamma_2 + \alpha_2 & -c_1\alpha_1(\xi_1)X^1 & \Gamma_0 & -d_1\alpha_1(\xi_1)X^1 \\ +d_1X^3) & +d_3X^3) & +0 & -c_1\alpha_1(\xi_3)X^3 & +0 & -d_1\alpha_1(\xi_3)X^3 \\ & & & -c_3\alpha_3(\xi_1)X^1 & & -d_3\alpha_3(\xi_1)X^1 \\ & & & -c_3\alpha_3(\xi_3)X^3) & & -d_3\alpha_3(\xi_3)X^3) \\ & & \Gamma_3 + \alpha_3 & & \Gamma_1 - \alpha_1 & \\ & & +\lambda^2(b_1\alpha_1 + b_3\alpha_3) & & +\lambda^2(d_1\alpha_1 + d_3\alpha_3) & \\ \lambda(b_1X^0 & \lambda(b_3X^0 & -b_1\alpha_1(\xi_1)X^1 & -\Gamma_2 + \alpha_2 & -d_1\alpha_1(\xi_1)X^1 & \Gamma_0 \\ +d_1X^2) & +d_3X^2) & -b_1\alpha_1(\xi_3)X^3 & +0 & -d_1\alpha_1(\xi_3)X^3 & +0 \\ & & -b_3\alpha_3(\xi_1)X^1 & & -d_3\alpha_3(\xi_1)X^1 & \\ & & -b_3\alpha_3(\xi_3)X^3) & & -d_3\alpha_3(\xi_3)X^3) & \end{pmatrix}$$

The skew-symmetric matrix Γ_λ^Z in (3-13) is the connection matrix of the Levi-Civita connection of the Z-metric g_λ^Z .

It follows from (3-13) and the first structure equation that dX^i ($i = 0, 1, 2, 3$) is expressed as the sum of terms of the form $X^i \wedge (\text{something})$ ($i = 0, 1, 2, 3$). Therefore the 2-dimensional distribution on \mathcal{Z} defined by the system of equations $X^i = 0$ ($i = 0, 1, 2, 3$) is integrable. Moreover this distribution is holomorphic w.r.to the canonical complex structure of \mathcal{Z} . The expression (3-13), the first structure equation and the formula $d(2\alpha_2) = 4\alpha_3 \wedge \alpha_1 + 4({}^tX^2 \wedge X^0 + {}^tX^3 \wedge X^1)$ imply that every integral submanifold of this distribution has constant curvature 4. Therefore any integral submanifold of the distribution $X^i = 0$ ($i = 0, 1, 2, 3$) is a holomorphic curve in \mathcal{Z} isomorphic to \mathbb{P}^1 and more precisely a deformation of the twistor line¹⁰.

The curvature form Ω_λ^Z is computed from the second structure equation $\Omega_\lambda^Z = d\Gamma_\lambda^Z + \Gamma_\lambda^Z \wedge \Gamma_\lambda^Z$. As (a_i, b_i, c_i, d_i) ($i = 1, 3$) are all “invisible” at $z \in \mathcal{Z}$, we conclude that the $\Gamma_\lambda^Z \wedge \Gamma_\lambda^Z$ -part of the

¹⁰ In the definition of $L'_{J(z)}$ we have chosen $1 \in H^0(\mathcal{P}^1, \mathcal{O})$. However, we can start with any element from $H^0(\mathbb{P}^1, \mathcal{O})$ and develop a similar theory and if we start with a constant from $H^0(\mathbb{P}^1, \mathcal{O})$ which is very small in absolute value, then the distribution \mathcal{D}_z should be very close to the horizontal distribution and therefore the resulting analogue of a Z-metric with $\lambda = 1$ should be very close to the basic canonical deformation metric. Therefore the integral manifold under consideration must be a deformation of a twistor line.

curvature form Ω_λ^Z is the same as the corresponding part of the curvature form of the original quaternion Kähler manifold (M, g) . On the other hand, we can compute the $d\Gamma_\lambda^Z$ -part in the following way. We first observe that if we impose the condition (3-2') on $\nabla\xi_0$, we conclude that the differentiation of the “invisible” terms gives rise to 1-forms such exactly like X^i , modulo α_1 and α_3 :

$$\begin{aligned} \alpha_1(\xi_0) &\xrightarrow{\text{differentiation}} X^1, \quad \alpha_1(\xi_2) \xrightarrow{\text{differentiation}} -X^3, \\ \alpha_3(\xi_0) &\xrightarrow{\text{differentiation}} X^3, \quad \alpha_3(\xi_2) \xrightarrow{\text{differentiation}} X^1. \end{aligned}$$

For instance, we have

$$\begin{aligned} d(\alpha_1(\xi_0)) &= (\nabla\alpha_1)(\xi_0) + \alpha_1(\nabla\xi_0) \\ &= (2\alpha_2 \otimes \alpha_3 - {}^tX^0 \otimes X^1 + {}^tX^1 \otimes X^0 + \cdots)(\cdot, \xi_0) \\ &\quad + \alpha_1(\Gamma_0 \otimes \xi_0 + (\Gamma_1 + \alpha_1) \otimes \xi_1 + (\Gamma_2 + \alpha_2) \otimes \xi_2 + (\Gamma_3 + \alpha_3) \otimes \xi_3) \\ &= {}^tX^1 \text{ modulo } \alpha_1 \text{ and } \alpha_3. \end{aligned}$$

ignoring “invisible” terms. We have used the condition (3-2') on the Γ -part in $\nabla\xi_0$. We can check other cases similarly. Secondly, we combine the above observation with the general formula

$$\text{Ric}(e_i, e_j) = \sum_{k=1}^{\dim} g(\Omega_k^j(e_i, e_k)e_j, e_j) .$$

We see that all terms in the connection matrix Γ_λ^Z whose origin stem from the $a_i(\xi_j)\Gamma_k \wedge X^l$ -part ($i = 1, 3$ and $j, k, l = 0, 2$) in the derivation formulae (3-11) and (3-12), i.e., all terms which appears with one of (a_i, b_i, c_i, d_i) 's ($i = 1, 3$), do not contribute to the Ricci tensor. This means that the Ricci form of the Z-metric g_λ^Z has the form of the sum of $\hat{\alpha}_1^2 + \hat{\alpha}_3^2$ and $\sum_{i=0}^3 {}^tX^i \cdot X^i$ with some coefficients. For the explicit computation, we introduce $\{\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1}, \xi_0, \xi_1, \xi_2, \xi_3\}$ which is the orthonormal frame dual to the orthonormal coframe $\{\lambda\hat{\alpha}_1, \lambda\hat{\alpha}_3, X^0, X^1, X^2, X^3\}$. As the first example, the non-trivial contribution to the Ricci tensor in the ξ_{-2} -direction comes from $d\alpha_2$. Recall that $d\alpha_2$ is given by the formula

$$d\alpha_2 = 2\alpha_3 \wedge \alpha_1 + 2({}^tX^2 \wedge X^0 + {}^tX^3 \wedge X^1) .$$

We have $X^j(\xi_i) = 0$ for $i < 0$ and $j \geq 0$ and moreover from the definition of $\hat{\alpha}_i$ we have $\hat{\alpha}_i \equiv \alpha_i$ modulo X^j ($i = 1, 3$ and $j \geq 0$). Therefore we have

$$\begin{aligned} \text{Ric}_\lambda^Z(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-2}) &= g_\lambda^Z((\Omega_\lambda^Z)_{-1}^{-2}(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1})\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-2}) \\ &= g_\lambda^Z((4\alpha_1 \wedge \alpha_3)(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1})\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-2}) = \frac{4}{\lambda^2} . \end{aligned}$$

As the second example, the non-trivial contribution to the Ricci tensor in the X^0 -direction comes from the curvature form of the original metric g :

$$\begin{aligned} \text{Ric}_\lambda^Z(\xi_0, \xi_0) &= g_\lambda^Z\left(\sum_{i=3,1} (\Omega_\lambda^Z)_{-i}^0(\xi_0, \lambda^{-1}\xi_i)\xi_0 + \sum_{k=1}^3 (\Omega_\lambda^Z)_k^0(\xi_0\xi_k)\xi_0, \xi_0\right) \\ &= g\left(\sum_{k=1}^3 \Omega_k^0(\xi_0, \xi_k)\xi_0, \xi_0\right) = \text{Ric}(\xi_0, \xi_0) = 4n + 8 . \end{aligned}$$

Similarly, we have

$$\begin{aligned} \text{Ric}_\lambda^Z(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-2}) &= \text{Ric}_\lambda^Z(\lambda^{-1}\xi_{-1}, \lambda^{-1}\xi_{-1}) = \frac{4}{\lambda^2}, \\ \text{Ric}_\lambda^Z(\xi_0, \xi_0) &= \text{Ric}_\lambda^Z(\xi_1, \xi_1) = \text{Ric}_\lambda^Z(\xi_2, \xi_2) = \text{Ric}_\lambda^Z(\xi_3, \xi_3) = 4n + 8 \\ \text{all other components of } \text{Ric}_\lambda^Z &= 0. \end{aligned}$$

Summing up the computations in §3, we have the following Proposition.

Proposition 3.1. *The Ricci tensor of the Z-metric*

$$g_\lambda^Z = \lambda^2(\hat{\alpha}_1^2 + \hat{\alpha}_3^2) + \sum_{i=0}^3 {}^tX^i \cdot X^i$$

on the twistor space \mathcal{Z} is given by the formula

$$(3-14) \quad \text{Ric}_\lambda^Z = \frac{4}{\lambda^2} \lambda^2(\hat{\alpha}_1^2 + \hat{\alpha}_3^2) + (4n + 8) \sum_{i=0}^3 {}^tX^i \cdot X^i.$$

In particular, the Z-metric g_λ^Z is Einstein if and only if $\lambda^2 = \frac{1}{n+2}$.

Proposition 2.4 implies that the family of canonical deformation metrics g_λ^{can} contains two Einstein metrics, i.e., those for $\lambda^2 = 1$ and $\lambda^2 = \frac{1}{n+1}$. From Proposition 3.1, the Z-metric g_λ^Z for $\lambda^2 = \frac{1}{n+2}$ turns out to be the third Einstein metric. This turns out to be non-isometric to any Einstein metric in the family of canonical deformation metrics.

Remark 3.2. The Einstein metric g_λ^Z with $\lambda^2 = \frac{1}{n+2}$ in the family of Z-metrics on \mathcal{Z} is not isometric to the Einstein metrics g_λ^{can} with $\lambda^2 = 1$ or $\lambda^2 = \frac{1}{n+1}$ in the family of canonical deformation metrics.

Proof. It is clear from the construction that, if (M, g) is a Wolf space, then both canonical deformation metrics g_λ^{can} and Z-metrics g_λ^Z are homogeneous w.r.to the action of the isometry group of (M, g) on the twistor space \mathcal{Z} . The assertion is then easy to check in the model case $\mathbb{P}^3(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{H})$. Indeed, the difference of these three Einstein metrics are visible in the relationship between the orthonormal basis of $T_z\mathcal{Z}$, the fixed complex line $L_{J(z)}$ (cf. discussion before (3-1)) in the horizontal subspace (of the twistor fibration $\mathbb{P}^3(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{H})$) and the tangent space of the twistor line \mathbb{P}^1 (the $\text{Sp}(1)$ -orbit). \square

Remark 3.3. (1) Homogeneous Einstein metrics on $\mathbb{P}^{2n+2}(\mathbb{C})$ were classified by Ziller [Z]. The canonical deformation metrics g_1^{can} and $g_{\frac{1}{n+1}}^{\text{can}}$ are the only homogeneous Einstein metrics on \mathbb{P}^{2n+1} up to homothety. Therefore the Z-metric $g_{\frac{1}{n+2}}^Z$ on the twistor space of $M = \mathbb{P}^n(\mathbb{H})$ is not a

$\text{Sp}(n+1)$ -homogeneous Einstein metric on the twistor space $\mathcal{Z} = \frac{\text{Sp}(n+1)}{\text{Sp}(n) \times \text{SO}(2)} = \mathbb{P}^{2n+1}(\mathbb{C})$.

The reason why g_λ^Z on $\mathbb{P}^{2n+1}(\mathbb{C})$ is not $\text{Sp}(n+1)$ -homogeneous can be seen from the $\text{Sp}(n+1)$ -orbit decomposition of the Grassmannian $\text{Grass}(\mathbb{C}^2, \mathbb{C}^{2n+2})$ of all \mathbb{P}^1 's (lines's) in \mathbb{P}^{2n+1} , which is the complexification of $\mathbb{P}^n(\mathbb{H})$. Indeed, there exists only one $(4n)$ -dimensional $\text{Sp}(n+1)$ -orbit, which is $\mathbb{P}^n(\mathbb{H})$ corresponding to the twistor fibration. It would be interesting to determine the group of isometries of $g_{\frac{1}{n+2}}^Z$. Is $g_{\frac{1}{n+2}}^Z$ of cohomogeneity one?

(2) The fiber of the $\mathrm{Sp}(1)$ -principal bundle $\tilde{\mathcal{Z}} \rightarrow M$ is the parameter space of all “framed” $L'_{J(z)}$ ’s (z lying on a fixed twistor line) and therefore the extended twistor space $\tilde{\mathcal{Z}}$ itself is identified with the space of all “framed” $L'_{J(z)}$ ($z \in \mathcal{Z}$). Since the distribution defined by $L'_{J(z)}$ is integrable¹¹, we can introduce an equivalent relation on $\tilde{\mathcal{Z}}$ where two points of $\tilde{\mathcal{Z}}$ are equivalent if and only if these lie on the same integral submanifold (extended by the S^1 -fiber of the fibration $\tilde{\mathcal{Z}} \rightarrow \mathcal{Z}$) of the distribution \mathcal{D}^\perp . The same argument works if we start with the distribution \mathcal{D}^\perp instead of the distribution defined by $L'_{J(z)}$ and therefore we can identify $\tilde{\mathcal{Z}}$ with the space of “framed” \mathcal{D}^\perp . The quotient space obtained from $\tilde{\mathcal{Z}}$ (moduli space of “framed” \mathcal{D}^\perp) and the above equivalence relation (defined by the integral submanifold of \mathcal{D}^\perp) turns out to be a realization of M . The resulting quotient map induces a \mathbb{P}^1 -fibration $\mathcal{Z} \rightarrow M$ different from the twistor fibration. This is a Riemannian submersion w.r.to any Z-metric on \mathcal{Z} and the original quaternion Kähler metric on M . However the fibers are not totally geodesic (see Remark 3.4).

Remark 3.4. In the case of the canonical deformation metrics, the formula $d\alpha_2 = 2\alpha_3 \wedge \alpha_1 + 2({}^tX^2 \wedge X^0 + {}^tX^3 \wedge X^1)$ implies that the vertical distribution is integrable and the twistor lines are totally geodesic. In the case of the Z-metrics, the formula $d\alpha_2 = 2\alpha_3 \wedge \alpha_1 + 2({}^tX^2 \wedge X^0 + {}^tX^3 \wedge X^1) = 2(\hat{\alpha}_3 + \alpha_3(\xi_i)X^i) \wedge \hat{\alpha}_1 + \alpha_1(\xi_j)X^j + ({}^tX^2 \wedge X^0 + {}^tX^3 \wedge X^1) = 2\hat{\alpha}_3 \wedge \hat{\alpha}_1 + \hat{\alpha}_3 \wedge \alpha_1(\xi_j)X^j + \alpha_3(\xi_i)X^i \wedge \hat{\alpha}_1 + \alpha_3(\xi_i)X^i \wedge \alpha_1(\xi_j)X^j + 2({}^tX^2 \wedge X^0 + {}^tX^3 \wedge X^1)$ implies that the distribution defined by the equations $X^i = 0$ ($i = 0, 1, 2, 3$) is integrable. This formula involves non-trivial mixed terms like $\alpha_3(\xi_i)X^i \wedge \hat{\alpha}_1$ and so on. Therefore the distribution defined by the equations $X^i = 0$ ($i = 0, 1, 2, 3$) is not totally geodesic.

Remark 3.5 (Comparison with the orbifold case). We can construct locally irreducible positive quaternion Kähler orbifolds which are uniformized by one of the Wolf spaces. On the other hand, many examples of non locally symmetric positive quaternion Kähler orbifolds are constructed in [G-L]. Here we remark that the moving frame construction of Z-metrics in §3 does not necessarily generalize to positive quaternion Kähler orbifold case. Here we explain the reason. If we take a local uniformization of the orbifold along the locus of orbifold singularities, we locally get a non-singular irreducible quaternion Kähler manifold with a finite group G acting isometrically preserving the local quaternion Kähler structure and therefore G operates on the local holonomy reduction $\mathcal{P}_{\mathrm{loc}}$ of the oriented orthonormal frame bundle. We can construct the orbifold version of the twistor space. To see what happens to the construction of the orbifold version of the Z-metrics, we work on the local uniformization level. The essential step in the construction is to determine the complex line in each \mathbb{H} -linear subspace explained in the beginning of §3. We recall this step. Let J be the orthogonal complex structure of \mathbb{H} represented by $z \in \mathbb{P}_m^1$. Then this defines a S^1 -subgroup in $\mathrm{Sp}(1) = \mathrm{Sp}(1)_r$ acting from the right on $T_m M$. Identifying $\mathrm{Sp}(1)_r$ with $\mathrm{Sp}(1)_l$ we get a S^1 subgroup of $\mathrm{Sp}(1)_l$. This S^1 -subgroup determines a complex line L_J in \mathbb{H} (corresponding to the axis of the rotation of the action induced on $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}_J^2)$). Suppose that $z \in \mathbb{Z}$ is fixed by a non-trivial subgroup of G . Then the linear isotropy representation defines a finite subgroup of $\mathrm{Sp}(1)$ acting on the \mathbb{H} -line. The orbifold version of Z-metrics is well-defined if and only if the association $\mathbb{P}_m^1 \ni z \mapsto L_J \in \mathbb{P}^1(\mathbb{C}_J^2)$ is preserved by the action of G . In other words, the complex line L_J is not necessarily fixed by this action and if not fixed, the very beginning of the construction of Z-metrics do not work. Therefore we cannot define Z-metric in the equivariant way and this

¹¹ We have shown that the distribution \mathcal{D}^\perp defined by $X^i = 0$ ($i = 0, 1, 2, 3$) is integrable (see the arguments after (3.13)). The statement follows because \mathcal{D}^\perp is of the same type as the distribution defined by $L'_{J(z)}$ and therefore the same reasoning works in the proof of the integrability.

implies that the orbifold Z-metric is not defined in general (the case where the orbifold version of the Z-metric is defined corresponds to orbifolds uniformized by the Wolf spaces).

§4. Ricci Flow on the Twistor Space of a Positive Quaternion Kähler Manifold.

In §4, we study the behavior of the canonical deformation metrics \mathcal{F}^{can} and the Z-metrics \mathcal{F}^{Z} on the twistor space \mathcal{Z} of a positive quaternion Kähler manifold (M, g) . Let (M^{4n}, g) ($n \geq 2$) be a compact quaternion Kähler manifold with positive scalar curvature. Then (M, g) is positive Einstein satisfying $\text{Ric}(g) = (4n + 8)g$. Then the solution of the Ricci flow equation $\partial_t g = -2\text{Ric}$ with the initial condition $g(-1) = (8n + 16)g$ at time $t = -1$ is given by the homothety $g(t) = -(8n + 16)t g$ ($-\infty < t < 0$). Indeed, putting $g(t) = \lambda(t)g$, the above initial value problem of the Ricci flow equation becomes $\lambda'(t) = -(8n + 16)$, $\lambda(-1) = 8n + 16$. Its solution is $\lambda(t) = -(8n + 16)t$. Therefore the metric g is just the fixed point modulo homothety of the Ricci flow and is not so interesting as itself. However, we have more freedom in the twistor space \mathcal{Z} . Indeed, as in §2 and §3, we can construct, from the original quaternion Kähler metric, two Einstein metrics g_1^{can} and g^{Z} on \mathcal{Z} and two families of metrics $\mathcal{F}^{\text{can}} = \{g_\lambda^{\text{can}}\}$ and $\mathcal{F}^{\text{Z}} = \{g_\lambda^{\text{Z}}\}$ on \mathcal{Z} containing one of these Einstein metrics.

Proposition 4.1. *The homothetically extended family $\{\rho g_\lambda^{\text{can}}\}_{\rho, \lambda > 0}$ of the canonical deformation metrics on the twistor space \mathcal{Z} consists of orbits of the Ricci flow, i.e., the Ricci flow equation preserves the family $\{\rho g_\lambda^{\text{can}}\}_{\rho, \lambda > 0}$ on \mathcal{Z} .*

Proof. The conclusion of Theorem 4.1 is a consequence from the following two facts. Firstly, the family $\{\rho g_\lambda^{\text{can}}\}_{\rho, \lambda > 0}$ is closed under the convex sum. Secondly, although the Ricci tensor $\text{Ric}_\lambda^{\text{can}}$ of g_λ is not necessarily positive definite, it is of the same type as the canonical deformation metrics on \mathcal{Z} . Indeed, we have from (2-15) in Proposition 2.4 the formula

$$\begin{aligned} \text{Ric}_\lambda^{\text{can}} &= 4(1 + n\lambda^4)(\alpha_1^2 + \alpha_3^2) + 4(n + 2 - \lambda^2) \sum_{i=0}^3 {}^t X^i \cdot X^i \\ &= 4(n + 2 - \lambda^2) g^{\text{can}}_{\sqrt{\frac{1+n\lambda^4}{n+2-\lambda^2}}} . \end{aligned}$$

Here we must assume that $\lambda^2 < n + 2$. Combining these two facts, we infer that the Ricci flow equation $\partial_t g = -2\text{Ric}_g$ preserves the family $\{\rho g_\lambda^{\text{can}}\}_{\rho, \lambda > 0}$ if $\lambda^2 < n + 2$. \square

Proposition 4.2 implies that the Ricci flow with initial metric chosen from the family of canonical deformation metrics reduces to a system of ODE's :

Proposition 4.2. *The Ricci flow equation $\partial_t g = -2\text{Ric}_g$ on the twistor space \mathcal{Z} with initial metric in the homothetically extended family of the canonical deformation metrics reduces to the system of ordinary differential equations*

$$(4-1) \quad \begin{cases} \frac{d}{dt}(\rho(t)\lambda^2(t)) = -8(1 + n\lambda(t)^4) , \\ \frac{d}{dt}\rho(t) = -8(n + 2 - \lambda(t)^2) . \end{cases}$$

We examine the behavior of the solutions using the equation

$$(4-2) \quad \rho \frac{d\lambda^2}{dt} = -4\{(n + 1)\lambda^2 - 1\}(\lambda^2 - 1) .$$

A solution to the system of ODE's (4-1) corresponds to a curve in (λ, ρ) -plane (where $\lambda, \rho > 0$). The curve $\lambda^2 = 1$ and $\lambda^2 = \frac{1}{n+1}$ correspond to two Einstein metrics. The solution with initial metric g_λ^{can} λ^2 slightly larger than 1 corresponds to a curve $(\lambda(t), \rho(t))$ with the property that both $\lambda(t)$ and $\rho(t)$ decrease as t increases. The right hand side of (4-2) is strictly negative if $\lambda^2 > 1$. This observation implies that the solution converges to the Kähler-Einstein metric g_1^{can} (modulo scaling). Similarly, the solution with initial metric g_λ^{can} with λ^2 slightly smaller than 1 corresponds to a curve $(\lambda(t), \rho(t))$ with the property that $\lambda(t)$ increases and $\rho(t)$ decreases as t increases. Moreover (4-2) implies that $\rho \frac{d\lambda^2}{dt} > 0$ which implies that $\rho > 0$ if $\frac{1}{n+1} < \lambda^2 < 1$. This implies that the solution converges to the Kähler-Einstein metric g_1^{can} (modulo scaling). Next we look at the solution with initial metric g_λ^{can} λ^2 being slightly larger than $\frac{1}{n+1}$. In this case λ increases as t increases. Therefore the solution approaches to the Einstein metric $g_{\sqrt{\frac{1}{n+1}}}^{\text{can}}$ (modulo scaling) as $t \rightarrow \infty$, i.e., the solution is an ancient solution. However, as t increases, the solution becomes extinct in finite time and after scaling approaches to the Kähler-Einstein metric $g_{\sqrt{\frac{1}{n+1}}}^{\text{can}}$. Similarly, the solution with initial metric g_λ^{can} λ^2 being slightly smaller than $\frac{1}{n+1}$ is also an ancient solution. To examine the behavior when t increases, we need the defining equation of the trajectory. Eliminating t from (4-1) we have $\log \rho - c = (1 + \frac{1}{n}) \log |\lambda^2 - 1| - \frac{1}{n+1} (n + \frac{1}{n} + 3) \log |(n+1)\lambda^2 - 1|$ for $\exists c \in \mathbb{R}$. It follows from this that the solution becomes totally singular in finite time and realizes the collapse corresponding to the limit $\lambda \rightarrow 0$ and $\rho \rightarrow e^c > 0$, i.e., \mathcal{Z} collapses to M . Because of these properties when t increases, the analysis in §5 does not apply in this situation.

We next look at the homothetically extended family of Z-metrics.

Proposition 4.3. *The homothetically extended family $\{\rho g_\lambda^Z\}_{\rho, \lambda > 0}$ of the Z-metrics on the twistor space \mathcal{Z} consists of orbits of the Ricci flow, i.e., the Ricci flow equation preserves the family $\{\rho g_\lambda^Z\}_{\rho, \lambda > 0}$ on \mathcal{Z} .*

Proof. The conclusion of Proposition 4.3 is a consequence from the following two facts. Firstly, the family $\{\rho g_{\lambda^Z}\}_{\rho, \lambda > 0}$ is closed under the convex sum. Secondly, the Ricci tensor Ric_λ^Z of g_λ is positive definite and is of the same type as Z-metrics s on \mathcal{Z} . Indeed, we have from (3-14) in Proposition 3.1 the formula

$$\text{Ric}_\lambda^Z = 4(\alpha_1^2 + \alpha_3^2) + (4n+8) \sum_{i=0}^3 {}^t X^i \cdot X^i = (4n+8) g_{\sqrt{\frac{1}{n+2}}}^Z.$$

Combining these two facts, we infer that the Ricci flow equation $\partial_t g = -2 \text{Ric}_g$ preserves the family $\{\rho g_\lambda^Z\}_{\rho, \lambda > 0}$. \square

Theorem 4.4. (1) *The Ricci flow equation $\partial_t g = -2 \text{Ric}_g$ on the twistor space \mathcal{Z} with initial metric in the homothetically extended family of the Z-metrics reduces to the system of ordinary differential equations*

$$(4-3) \quad \begin{cases} \frac{d}{dt}(\rho(t)\lambda^2(t)) = -8, \\ \frac{d}{dt}\rho(t) = -8(n+2). \end{cases}$$

(2) *For any initial metric with $\lambda^2 = \frac{1}{n+2}$ at time $t = 0$ in the homothetically extended family of Z-metrics on \mathcal{Z} , the system of ODE's (4-2) has a solution*

$$\lambda^2 \equiv \frac{1}{n+2}.$$

This corresponds to the Einstein metric $g^Z_{\sqrt{\frac{1}{n+2}}}$.

(3) For any initial metric $\rho_0 g^Z_{\lambda_0}$ with $\lambda_0^2 \neq \frac{1}{n+2}$ at time $t = 0$ in the homothetically extended family of Z -metrics on \mathcal{Z} , the system of ODE's (4-3) has a solution

$$\begin{aligned}\rho(t) &= \rho_0 - 8(n+2)t, \\ \rho\lambda^2(t) &= \rho_0\lambda_0^2 - 8t.\end{aligned}$$

This implies that the Ricci flow solution $g(t)$ is given by

$$g(t) = (\rho_0\lambda_0^2 - 8t)(\hat{\alpha}_1^2 + \hat{\alpha}_3^2) + (\rho_0 - 8(n+2)t) \sum_{i=0}^3 {}^tX^i \cdot X^i.$$

The solution is an ancient solution. Moreover its asymptotic soliton is the Einstein metric g^Z_{λ} with $\lambda^2 = \frac{1}{n+2}$, i.e., modulo scaling, the solution $g(t)$ converges to the Einstein metric $g^Z_{\sqrt{\frac{1}{n+2}}}$ as $t \rightarrow -\infty$.

If $\lambda_0^2 > \frac{1}{n+2}$ at $t = 0$, then the solution becomes extinct in finite time and after scaling approaches to a Carnot-Carathéodory metric corresponding to g^Z_{λ} with $\lambda \rightarrow \infty$.

If $\lambda_0^2 < \frac{1}{n+2}$ at $t = 0$, then the solution becomes totally singular in finite time and after scaling realizes the “collapse” of \mathcal{Z} corresponding to $\lambda \rightarrow 0$ and $\rho \rightarrow \rho_0\{1 - (n+2)\lambda_0^2\} > 0$.

Proof. The assertions (1), (2) and (3) follow from Proposition 4.3. On the other hand, the solution to (4-3) described in (3) implies

$$(4-4) \quad \lambda^2(t) = \frac{\rho_0\lambda_0^2 - 8t}{\rho_0 - 8(n+2)t}$$

from which the rest of the assertion (3) follows. \square

The remarkable difference from Proposition 4.3 is the following, which explains the meaning of Theorem 4.4 (3). A solution of (4-3) with λ_0^2 slightly larger than $\frac{1}{n+1}$ is an ancient solution having the Einstein metric $g^{\text{can}}_{\sqrt{\frac{1}{n+1}}}$ as its asymptotic soliton when $t \rightarrow -\infty$ and it approximates the Kähler-Einstein metric g^{can}_1 just before the extinction (after scaling). A solution of (4-3) with λ^2 slightly larger than $\frac{1}{n+2}$ is also an ancient solution having the Einstein metric $g^Z_{\sqrt{\frac{1}{n+2}}}$ as its asymptotic soliton. However, the behavior of the solution just before the extinction is essentially different. Namely we can scale the solution so that the \mathcal{D} -direction survives in the limit toward the extinction time. We will exploit this difference in §5.

Eliminating t from (4-3) we have

$$(4-5) \quad \rho = \frac{\rho_0 \left(\lambda_0^2 - \frac{1}{n+2} \right)}{\lambda^2 - \frac{1}{n+2}}.$$

This is the equation of the trajectory of the Ricci flow solution with initial metric corresponding to (λ_0, ρ_0) .

The 2-dimensional family $\mathcal{F} = \{\rho g_{\lambda}\}_{\rho>0, \lambda>0}$ constitutes a Ricci flow unstable cell in the sense that the family \mathcal{F} is foliated by the trajectories of the Ricci flow solutions and each Ricci flow trajectory is an ancient solution whose asymptotic soliton (in the sense of [P, §11]) corresponds to a Kähler-Einstein metric.

Example 4.5. Pick a trajectory defined by the equation

$$\rho = \frac{\rho_0 \left(\lambda_0^2 - \frac{1}{n+2} \right)}{\lambda^2 - \frac{1}{n+2}}$$

where $\rho_0 > 0$ and $\lambda_0^2 > \frac{1}{n+2}$ in the (λ, ρ) -plane identified with the family $\mathbb{R}_+ \cdot \mathcal{F}^Z$. Along this trajectory we have $\text{Ric}_\lambda^Z = 4(\alpha_1^2 + \alpha_3^2) + (4n+8) \sum_{i=0}^3 {}^t X^i \cdot X^i = (4n+8) g_{\sqrt{\frac{1}{n+2}}}^Z$. Therefore the scalar curvature along the trajectory is

$$\text{Scal}(\rho g_\lambda^Z) = \frac{8}{\rho \lambda^2} + \frac{16(n+2)n}{\rho}.$$

If we set $u = \text{constant}$ determined by $\int_M u dV = 1$, i.e., $u = 1/\text{Vol}(g_{ij}(t))$, $g_{ij}(t)$ being the solution, we get a solution $u(t, x)$ (t -dependent constant function on M) to the conjugate heat equation

$$\partial_t u = -\Delta u + Ru. \text{ Since } \text{Vol}(\rho g_\lambda^Z) = \frac{\rho_0^{2n+1} \left(\lambda_0^2 - \frac{1}{n+2} \right)^{2n+1}}{\left(\lambda^2 - \frac{1}{n+2} \right)^{2n+1}} \lambda^2 \text{Vol}(M, g), \text{ we have}$$

$$u = \frac{\left(\lambda^2 - \frac{1}{n+2} \right)^{2n+1}}{\rho_0^{2n+1} \left(\lambda_0^2 - \frac{1}{n+2} \right)^{2n+1}}.$$

Eliminating ρ from (4-3) we have

$$\tau := -t = \frac{1}{8(n+2)} \left\{ \frac{\rho_0 \left(\lambda_0^2 - \frac{1}{n+2} \right)}{\lambda^2 - \frac{1}{n+2}} - \rho_0 \right\}.$$

We observe that

$$\lambda^2 \rightarrow \frac{1}{n+2} + 0 \iff \tau \rightarrow \infty \iff t \rightarrow -\infty.$$

The function $W(g_{ij}, f, \tau)$ (W being Perelman's W -functional) is monotone increasing along the Ricci flow trajectory passing through a metric $\rho_0 g_{\lambda_0}^Z$ with $\lambda_0^2 > \frac{1}{n+2}$, which is determined by the triple $(\rho g_\lambda, f, \tau)$ where ρ, λ, τ are given as above, $\lambda^2 \in (\frac{1}{n+2}, \infty)$ increases to from $\frac{1}{n+2}$ to ∞ when τ decreases from ∞ to 0), and f is determined by setting $u = (4\pi\tau)^{-(2n+1)} e^{-f}$ with u and τ given as above.

§5. Uniformization of Positive Quaternion Kähler Manifolds.

Let M^{4n} be a compact quaternion Kähler manifold with positive scalar curvature and \mathcal{Z} its twistor space. Let $\mathbb{R}_+ \cdot \mathcal{F}^Z = \{\rho g_\lambda^Z\}_{\rho, \lambda > 0}$ be the homothetically extended family of Z -metrics on the twistor space \mathcal{Z} . Consider the ‘‘Ricci map’’ from the space of Riemannian metrics on \mathcal{Z} to the

space of symmetric $(0, 2)$ -tensors on \mathcal{Z} defined by $g \mapsto \text{Ric}_g$. Then we have shown in Section 3 and 4 (see Theorem 4.4) that

1) the space of homothetically extended family of Z-metrics $\mathbb{R}_+ \cdot \mathcal{F}^Z$ contains the half-line consisting the scalings of the Einstein metric $g^Z_{\sqrt{\frac{1}{n+2}}}$ on \mathcal{Z} .

2) The space of homothetically extended family of Z-metrics $\mathbb{R}_+ \cdot \mathcal{F}^Z$ on \mathcal{Z} consists of orbits of the Ricci flow, more precisely, the family $\mathbb{R}_+ \cdot \mathcal{F}^Z$ is invariant under the "Ricci map" and the Ricci flow.

It is natural to regard the existence of a special family of ancient solutions whose asymptotic soliton is a homothetical family of a fixed Einstein metric as an extension of the notion of a single Einstein metric. The following Theorem 5.1 is an evidence for the usefulness of such an extended notion. Indeed, we obtain strong information on the Einstein metric from the analysis of the ancient solutions. Our strategy is to apply Bando-Shi's gradient estimate ([B], [Sh1,2]) for the Ricci flow to the ancient solution in Theorem 4.4 (3) for $\lambda_0^Z > \frac{1}{n+2}$. To do so, we need to know the behavior of the full curvature tensor along the solution. Using the expression (3-13) of the Levi-Civita connection Γ_λ^Z of g_λ^Z and the second structure equation, we can estimate the norm of the curvature tensor of g_λ^Z . The second structure equation says that the curvature form is defined by $\Omega_\lambda^Z := d\Gamma_\lambda^Z + \Gamma_\lambda^Z \wedge \Gamma_\lambda^Z$. As a_i, \dots, d_i 's ($i = 1, 3$) are all "invisible" at $z \in \mathcal{Z}$, we have only to examine $d\Gamma_\lambda^Z$. We now estimate the norm of the curvature tensor by directly estimating all entries of the $d\Gamma_\lambda^Z$ -part of the curvature form. The reason we must do so is the following. Although estimating all sectional curvatures is equivalent to estimating the norm of the curvature tensor, estimating all sectional curvatures of the form $K(e_i, e_j)$ where $\{e_i\}$ is the orthonormal basis under consideration is not enough to estimate the norm of the curvature tensor. Now we look at (3-13). The trouble would be that the uncontrollable quantities of size λ or λ^2 appear in the curvature form as λ becomes large. So we estimate the norm of the curvature form modulo $O(1)$ when λ becomes large. The only trouble from this view point which may occur when λ becomes large stems from the exterior differential of the term like $\lambda^2(-a_1\alpha_1 - a_3\alpha_3 + a_1\alpha_1(\xi_1)X^1 + a_1\alpha_1(\xi_3)X^3 + a_3\alpha_3(\xi_1)X^1 + a_3\alpha_3(\xi_3)X^3)$ and so on. The definition of $\hat{\alpha}_i$ ($i = 1, 3$) implies that the above quantity is equal to $\lambda^2(-a_1\hat{\alpha}_1 - a_3\hat{\alpha}_3 - a_1\alpha_1(\xi_0)X^0 - a_1\alpha_1(\xi_2)X^2 - a_3\alpha_3(\xi_0)X^0 - a_3(\xi_2)X^2)$. Here the terms like $a_1\alpha_1(\xi_0)$ are a product of terms "invisible" at z and so on. Therefore, we have only to estimate the exterior differential of the quantity $a_1\hat{\alpha}_1$ and $a_3\hat{\alpha}_3$ to estimate the norm of the exterior differential of the original quantity. As a_1 and a_3 are "invisible" at z , or, as $d\hat{\alpha}_i$ ($i = 1, 3$) are "invisible" at z , we have only to consider $da_i \wedge \alpha_i$ ($i = 1, 3$). The condition (3-2') implies that the exterior differential da_1 consists of X^i modulo α_1 and α_3 . Therefore the exterior differential of $a_1\hat{\alpha}_1$ and $a_3\hat{\alpha}_3$ consists of $\hat{\alpha}_1 \wedge \hat{\alpha}_3$ and $X^i \wedge \hat{\alpha}_j$ ($i, j = 1, 3$). Note that the norm w.r.to the metric g_λ^Z of $\lambda^2\hat{\alpha}_1 \wedge \hat{\alpha}_3$ (resp. $\lambda^2 X^i \wedge \hat{\alpha}_j$) is 1 (resp. λ). The same is true for those quantity like $\lambda^2(-b_1\alpha_1 - b_3\alpha_3 + b_1\alpha_1(\xi_1)X^1 + b_1\alpha_1(\xi_3)X^3 + b_3\alpha_3(\xi_1)X^1 + b_3\alpha_3(\xi_3)X^3)$ and so on. By gathering all these terms with norm λ arising this way, we can specify a special part of the curvature tensor. The sum of the norms of this special part is λ times some constant (depending only on n).

Theorem 5.1. *Let (M^{4n}, g) be a compact quaternion Kähler manifold with positive scalar curvature. Then the family $\{g_\lambda^Z\}_{\lambda>1}$ of Z-metrics on the twistor space of M satisfies the limit formula*

$$\lim_{\lambda \rightarrow \infty} |\nabla^{g_\lambda^Z} \text{Rm}^{g_\lambda^Z}|_{g_\lambda^Z} = 0.$$

Proof. We follow the proof of Bando-Shi's derivative estimate for the curvature tensor under the Ricci flow ([B], [Sh1,2], see also [C-K, Chapt.7]). We consider a sequence $\{\lambda_k, \rho_k\}_{k=1}^\infty$ of pairs

of positive numbers satisfying $\forall \rho_k = 1$, $1 < \forall \lambda_k$ and $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$. This represents a sequence $\{g_{\lambda_k}^Z\}_{k=1}^\infty$ of the Z-metrics on the twistor space \mathcal{Z} . Pick one $g_{\lambda_k}^Z$ and write it as g_k . We consider the trajectory of the Ricci flow passing through g_k (with the maximal time interval in the past and future) and we write $g_k(t)$ for the Ricci flow solution where the initial metric $g_k(0)$ is taken from the trajectory as $\rho_k g_{\lambda(0)}^Z$ (ρ_k being large) such that $\lambda(0)$ is slightly larger than 1, say, $\lambda^2(0) = \frac{1}{n+2} + \delta$, and T_k is the time when the Ricci flow trajectory passes through the given $g_{\lambda_k}^Z := g_k : g_k(T_k) = g_k$. Let ∇ and Rm the Levi-Civita connection and the Riemann curvature tensor of the Ricci flow solution $g_k(t)$. Let $A * B$ denote any quadratic quantity obtained from $A \otimes B$ whose meaning is explained in [C-K, p. 227]. In the following computation, the norm at time t should be understood to be computed with respect to the metric $g_k(t)$. We have

$$(5-1) \quad \frac{\partial}{\partial t} |\text{Rm}|^2 = \Delta |\text{Rm}|^2 - 2|\nabla \text{Rm}|^2 + (\text{Rm})^{*3}$$

and

$$(5-2) \quad \frac{\partial}{\partial t} |\nabla \text{Rm}|^2 = \Delta |\nabla \text{Rm}|^2 - 2|\nabla^2 \text{Rm}|^2 + \text{Rm} * (\nabla \text{Rm})^{*2}.$$

The strategy of Bando-Shi derivative estimate is to make the best use of the good term $-2|\nabla \text{Rm}|^2$ in (5-1) to kill the bad term $\text{Rm} * (\nabla \text{Rm})^{*2}$ in (5.2). We also use (5-1) and (5-2) but in somewhat different way because in our case we can use the special properties of the curvature tensor of g_λ^Z . We have specified the special part of the curvature tensor of g_λ^Z . This was characterized by $R_{\eta\mu\nu\kappa}$ where only one index is negative (i.e., -1 or -2) and satisfies $|R_{\eta\mu\nu\kappa}| = \lambda$. Here, we are considering components of the curvature tensor w.r.to the orthonormal coframe $(\eta\hat{\alpha}_1, \lambda\hat{\alpha}_3, X^0, X^1, X^2, X^3)$. Let $\widetilde{\text{Rm}}$ denote the part of the curvature tensor obtained by subtracting the special part specified above, i.e., if $R_{\eta\mu\nu\kappa}$ is a component contained in $\widetilde{\text{Rm}}$ then the indices contain no negative number or more than two indices are negative (in fact the components with at least three indices are negative are 0). The advantage of introducing $\widetilde{\text{Rm}}$ is the following. If we replace Rm by $\widetilde{\text{Rm}}$ in (5-1), the first two terms in the RHS of (5-1) do not change, i.e., $\Delta |\text{Rm}|^2 = \Delta |\widetilde{\text{Rm}}|^2$ and $|\nabla \text{Rm}|^2 = |\nabla \widetilde{\text{Rm}}|^2$ hold. Indeed, $|\text{Rm}|^2$ and $|\widetilde{\text{Rm}}|^2$ differ only by a constant and $\Delta |\text{Rm}|^2 = \Delta |\widetilde{\text{Rm}}|^2$ follows. On the other hand, the portion of ∇Rm stemming from the above specified part of the curvature tensor vanish and therefore $\nabla \text{Rm} = \nabla \widetilde{\text{Rm}}$ follows. The third term $(\text{Rm})^{*3}$ contains all terms in the direct computation of $\partial/\partial t |\widetilde{\text{Rm}}|^2$ obtained by differentiating the metric tensor w.r.to t . It therefore follows from the Ricci flow equation and Proposition 4.3 that the term corresponding to the third term in the RHS of (5-1) in the computation of $\partial/\partial t |\widetilde{\text{Rm}}|^2$ is not larger than $c_n |\widetilde{\text{Rm}}|^2$ where c_n is a positive constant depending only on n . Therefore we have

$$(5-1') \quad \frac{\partial}{\partial t} (e^{-c_n t} |\widetilde{\text{Rm}}|^2) \leq \Delta (e^{-c_n t} |\widetilde{\text{Rm}}|^2) - 2e^{-c_n t} |\nabla \widetilde{\text{Rm}}|^2.$$

Moreover, if we replace Rm by $\widetilde{\text{Rm}}$ in (5-2), the first two terms in the RHS of (5-1) do not change because $\nabla \text{Rm} = \nabla \widetilde{\text{Rm}}$. The third term $\text{Rm} * (\nabla \text{Rm})^{*2}$ contains all terms in the direct computation of $\partial/\partial t |\nabla \widetilde{\text{Rm}}|^2$ obtained by differentiating the metric tensor w.r.to t . Therefore, It follows from the same reason as above that the term corresponding to the third term in the RHS of (5-2) in the computation of $\partial/\partial t |\nabla \widetilde{\text{Rm}}|^2$ is not larger than $c_n |\nabla \widetilde{\text{Rm}}|^2$ where c_n is the same positive constant as above. Therefore we have

$$(5-2') \quad \frac{\partial}{\partial t} (e^{-c_n t} |\nabla \widetilde{\text{Rm}}|^2) \leq \Delta (e^{-c_n t} |\nabla \widetilde{\text{Rm}}|^2) - 2e^{-c_n t} |\nabla^2 \widetilde{\text{Rm}}|^2.$$

To prove Theorem 5.1, suppose the contrary, i.e., the maximum over \mathcal{Z} of $|\widetilde{\nabla \text{Rm}}|^2 = |\nabla \text{Rm}|^2$ at time T_k is uniformly (w.r.to k) bounded below by a positive constant D . Note that $|\widetilde{\text{Rm}}|^2$ is uniformly bounded below by a positive constant for all T_k . This is because g is a positive Einstein metric on M and it follows from the second structure equation and (3-13) that the curvature tensor of M directly contributes to the norm of Ω_λ^Z . Thus we can conclude that there exists a positive constant C such that

$$(5-3) \quad 2\varepsilon |\widetilde{\nabla \text{Rm}}|^2 \geq \varepsilon C (|\widetilde{\nabla \text{Rm}}|^2 + \varepsilon |\widetilde{\text{Rm}}|^2)$$

holds at the point where $|\nabla \text{Rm}|$ takes its maximum. If $\varepsilon \ll 1$ then $C \approx 2$. Take a small positive constant ε to be determined later. From (5-1'), (5-2') and (5-3) we have

$$\begin{aligned} \frac{\partial}{\partial t} \{e^{-c_n t} (|\widetilde{\nabla \text{Rm}}|^2 + \varepsilon |\widetilde{\text{Rm}}|^2)\} &\leq -C e^{-c_n t} (|\widetilde{\nabla \text{Rm}}|^2 + \varepsilon |\widetilde{\text{Rm}}|^2) - 2 e^{-c_n t} |\nabla^2 \widetilde{\text{Rm}}|^2 \\ &\quad + (\text{error term depending on } \lambda \text{ and } \varepsilon) \end{aligned}$$

at the point where $|\widetilde{\nabla \text{Rm}}|$ takes its maximum. Here, the error term stems from the quantity which bound the Laplacian term $\Delta \{(e^{-c_n t} (|\widetilde{\nabla \text{Rm}}|^2 + \varepsilon |\widetilde{\text{Rm}}|^2))\}$ from above. We note that $\Delta |\text{Rm}|^2$ involves the curvature tensor of the metric g_k , we should take ε small compared to $\lambda = \lambda_k$ so that the Laplacian term $\Delta \{(e^{-c_n t} (|\widetilde{\nabla \text{Rm}}|^2 + \varepsilon |\widetilde{\text{Rm}}|^2))\}$ is uniformly of order, say, $\sqrt{\varepsilon}$. For this purpose we should take ε comparable to or smaller than $1/\lambda_k^2$. So, there exists a positive constant K , which is taken to be uniform for all $\varepsilon > 0$ sufficiently small in the sense that $\varepsilon \approx O(1/\lambda_k^2)$ holds, which satisfies the estimate

$$(5-4) \quad \frac{\partial}{\partial t} \{e^{-c_n t} (|\widetilde{\nabla \text{Rm}}|^2 + \varepsilon |\widetilde{\text{Rm}}|^2) - K \sqrt{\varepsilon} t\} \leq -\varepsilon C e^{-c_n t} (|\widetilde{\nabla \text{Rm}}|^2 + \varepsilon |\widetilde{\text{Rm}}|^2 - K \sqrt{\varepsilon} t) - 2 e^{-c_n t} |\nabla^2 \widetilde{\text{Rm}}|^2$$

at the point where $|\widetilde{\nabla \text{Rm}}|^2$ takes its maximum. The parabolic maximum principle implies that we have

$$(5-5) \quad \begin{aligned} \{e^{-c_n t} (|\widetilde{\nabla \text{Rm}}|^2 + \varepsilon |\widetilde{\text{Rm}}|^2 - K \sqrt{\varepsilon} t)\}_{t=T} &\leq \{e^{-c_n t} (|\widetilde{\nabla \text{Rm}}|^2 + \varepsilon |\widetilde{\text{Rm}}|^2 - K \sqrt{\varepsilon} t)\}_{t=0} \exp(-\varepsilon C T) \\ &\leq \frac{\text{const.}}{\rho_0^3} \exp(-\varepsilon C T) \end{aligned}$$

for $0 \leq \forall T \leq T_k$. From (5-5) we have

$$(5-6) \quad |\widetilde{\nabla \text{Rm}}|^2(T) + \varepsilon |\widetilde{\text{Rm}}|^2(T) - K \sqrt{\varepsilon} T \leq \frac{\text{const.}}{\rho_0^3} \exp\{(c_n - \varepsilon C)T\}.$$

Now we take $\varepsilon > 0$ so that

$$K \sqrt{\varepsilon} T \leq \frac{1}{2} D$$

holds (D being the assumed uniform lower bound of $\max_{z \in \mathcal{Z}} |\nabla \text{Rm}|^2$ when $k \rightarrow \infty$). Once we choose $\varepsilon > 0$ this way, (5-6) implies that, in the above discussion, we can replace c_n by $c_n - \varepsilon C$ and repeat the same argument. Thus we can repeat the same argument with c_n replaced by

$c_n - \varepsilon C, c_n - 2\varepsilon C, \dots$. There exists a positive integer N such that $c_n - N\varepsilon C < 0$. Therefore we have

$$(5-7) \quad |\widetilde{\nabla \text{Rm}}|^2(T) + \varepsilon |\widetilde{\text{Rm}}|^2(T) - K\sqrt{\varepsilon}T \leq \frac{\text{const.}}{\rho_0^3} \exp\{\underbrace{(c_n - N\varepsilon C)T}_{<0}\}.$$

For instance, we choose $\varepsilon \leq \frac{D^2}{4K^2T_k^2}$ and $N > \frac{8c_n K^2 T^2}{D^2}$. Then from (5-7) we have

$$(5-8) \quad |\widetilde{\nabla \text{Rm}}|^2(T_k) + O(T_k^{-2}) \leq \frac{\text{const.}}{\rho_0^3} \exp(-T_k).$$

As we have assumed that $\max_{z \in \mathcal{Z}} |\widetilde{\nabla \text{Rm}}|^2(T_k)$ is uniformly (w.r.to k) bounded below by a positive constant D , (5-8) implies

$$D \leq |\widetilde{\nabla \text{Rm}}|^2(T_k) + O(T_k^{-2}) \leq \frac{\text{const.}}{\rho_0^3} \exp(-T_k)$$

as $k \rightarrow \infty$ ($T_k \rightarrow \infty$), which is clearly a contradiction. Therefore

$$\lim_{k \rightarrow \infty} \max_{z \in \mathcal{Z}} |\nabla^{g_{\lambda_k}^Z} \text{Rm}(g_{\lambda_k}^Z)|_{g_{\lambda_k}^Z}(z, T_k) = 0$$

must be the case and the assertion of Theorem 5.1 follows. \square

Remark 5.2. (1) Although the proof of Theorem 5.1 is based on the local computation in §3 and §4, Theorem 5.1 is a global result. Indeed, the parabolic maximum principle, which does not make sense (and not correct) in local situation, played an essential role in its proof.

(2) The 2-parameter family $\{\rho g_{\lambda}^Z\}_{\rho, \lambda > 0}$ of homothetically extended Z -metrics on the twistor space \mathcal{Z} is foliated by the trajectories of the Ricci flow solution. We have two facts (i) each Ricci flow solution in this family is an ancient solution (in the sense of [H, §19]) and (ii) it realizes the “collapse” where the base $(\{\xi_i\}_{i=0}^3)$ direction shrinks faster when t becomes large. These two facts are essential in the proof of Theorem 5.1. Indeed, because each trajectory corresponds to an ancient solution, for any sequence λ_k such that $\lim_{k \rightarrow \infty} \lambda_k = \infty$, we can find a Ricci flow solution defined on $[0, T_k]$ with initial metric with $\lambda^2 = 1 + \delta$ and $g(T_k) = g_k$, where $g_k = g_{\lambda_k}$. This argument does not work for the ancient solution in Proposition 4.2 having the Einstein metric $g_{\sqrt{\frac{1}{n+1}}}^{\text{can}}$ as its asymptotic soliton.

Theorem 5.3. *Any locally irreducible positive quaternion Kähler manifold (M^{4n}, g) is isometric to one of the Wolf spaces.*

Proof. The goal is to prove $\nabla^g \text{Rm}(g) \equiv 0$, i.e., (M, g) is a Riemannian symmetric space. In order to do so we compute the covariant derivative $\nabla^{g_{\lambda}^Z} \text{Rm}(g_{\lambda}^Z)$ (we will write this simply as $\nabla \text{Rm}_{\lambda}^Z$ for brevity) using (3-13) and compare the result with the limit formula in Theorem 5.1 (note that the curvature form of g is contained in the curvature form of g_{λ}^Z). The covariant derivative $\nabla \text{Rm}_{\lambda}^Z$ is computed from $d\Omega_{\lambda}^Z$ and the product of Γ_{λ}^Z and Ω_{λ}^Z . From the second structure equation (or the second Bianchi identity) $d\Omega_{\lambda}^Z$ is equal to

$$(*) := d\Gamma_{\lambda}^Z \wedge \Gamma_{\lambda}^Z - \Gamma_{\lambda}^Z \wedge d\Gamma_{\lambda}^Z.$$

On the other hand Γ_λ^Z is “invisible” at the point $z \in \mathcal{Z}$ where the computation in §3 is performed, except for the Levi-Civita connection form of g . We use the matrix (3-13) of Γ_λ^Z to list the parts of $d\Gamma_\lambda^Z$ which do not come from the Levi-Civita connection form of g :

(i) $d\alpha_2 = 2\alpha_3 \wedge \alpha_1 + 2(X^2 \wedge X^0 + X^3 \wedge X^1)$. In the product $(*)$ this is coupled with the invisible term (i.e., α_2) and therefore it does not survive in $(*)$.

(ii) the 2-forms obtained by taking the exterior differential of $\lambda(a_1X^1 + b_1X^3)$ and so on in the first two rows of (3-13). These consist of the linear combination of $\lambda\alpha_i \wedge X^j$ and $\lambda X^i \wedge X^j$ ($i, j = 1, 3$) at z .

(iii) the 2-forms obtained by taking the exterior differential of $\lambda^2(-a_1\alpha_1 - a_3\alpha_3 + a_1\alpha_1(\xi_1)X^1 + a_1\alpha_1(\xi_3)X^3 + a_3\alpha_3(\xi_1)X^1 + a_3\alpha_3(\xi_3)X^3) = \lambda^2(-a_1\hat{\alpha}_1 - a_3\hat{\alpha}_3 - a_1\alpha_1(\xi_0)X^0 - a_1\alpha_1(\xi_2)X^2 - a_3\alpha_3(\xi_0)X^0 - a_3(\xi_2)X^2)$ and so on in (3-13). These consist of the linear combination of $\lambda^2X^i \wedge \hat{\alpha}_j$ and $\lambda^2\alpha_i \wedge \hat{\alpha}_j$ ($i, j = 1, 3$) at z .

In the computation of $\nabla\text{Rm}_\lambda^Z$, the 3-forms appearing from different entries of the matrix representation of $(*)$ are coupled with the tensor product of the members of the orthonormal coframe $(\hat{\alpha}_1, \hat{\alpha}_3, X^0, X^1, X^2, X^3)$ which are mutually orthogonal. Therefore, (ii) and (iii) from the above listed 2-forms contribute with coefficients λ and λ^2 (if they survive after computing $(*)$). Therefore, ignoring the “invisible” terms, we have the expansion

$$|\nabla\text{Rm}_\lambda^Z|^2 = A_0 + \lambda A_1 + \lambda^2 A_2 .$$

The limit formula in Theorem 5.1 implies $A_1 \equiv A_2 \equiv 0$. The A_0 -part consists only of the curvature form of the Levi-Civita connection of g . Therefore $A_0 = |\nabla^g\text{Rm}(g)|_g \equiv 0$ which completes the proof of Theorem 5.3. \square

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